Homology products on $\mathbb{Z}_2$-quotients of free loop spaces of spheres

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Introduction

Given a Riemannian manifold (M, g) one can ask if and how many closed geodesics of g on M exist. Here a closed geodesic is a nonconstant curve $c : [a, b] \to M$ that is a geodesic segment, i.e. $\nabla^g_{\dot{c}(t)}\dot{c}(t) = 0$ for all $t \in [a, b]$, and satisfies $c(a) = c(b)$, $\dot{c}(a) = \dot{c}(b)$. Here $\nabla^g_{\dot{c}(t)}$ denotes the covariant derivative along c induced by the Levi-Civita connection ∇^g of g.
Our guiding example is the sphere $S^n \subset \mathbb{R}^{n+1}$ with its standard metric g_{st}, that is the metric induced by the Euclidean metric of the ambient space $\mathbb{R}^{n+1}$. For $(M, g) = (S^n, g_{st})$ we know the answer to questions above: There are infinitely many closed geodesics, namely all the great circles when parametrized proportionally to arc length. If c is such a great circle which is in addition injective on (a, b) (i.e. a simple curve) it is a so-called prime geodesic. A prime closed geodesic is a geodesic (not necessarily simple) that is not a multiply iterated shorter closed geodesic. On (S^n, g_{st}) there are already infinitely many prime closed geodesic and each iteration of these adds another infinitely many to the count.
Despite this vast abundance of closed geodesic for the standard metric on S^n, it is yet not known whether an arbitrary Riemannian metric g on the sphere always has infinitely many geometrically distinct closed geodesics. The words "geometrically distinct" are important here, since given a closed geodesic c, then its r-fold iterate c^r, with $c^r(t) := c(rt)$, is another, different geodesic. It is known that there are Finsler metrics on S^n with only finitely many geometrically distinct closed geodesics (Katok examples, see [**Zil83**]). On the contrary, on S^2 any Riemannian metric has infinitely many geometrically distinct closed geodesics ([**Hin93**]), but on S^n with $n \geq 3$ it is not known whether every Riemannian metric has infinitely many closed geodesics.
If we restrict attention to compact smooth manifolds M, such as the sphere, then it is known that there is at least one closed geodesic for an arbitrary Riemannian metric ([**Kli82**, Theorem 2.4.20]). It is also known that "most" Riemannian metrics on a compact M have infinitely many closed geodesics. Of course we have to clarify what "most" means. Results in this direction are Theorems 9.8, 9.10 and 9.11 in [**Rad92**].

Let M be a compact smooth manifold. One method of addressing questions about closed geodesic is to view them as points of a space of closed curves and then study this space of curves: A closed geodesic c on M is in particular a loop, that is $c : [a, b] \to M$ is continuous and $c(a) = c(b)$. By linear reparametrization we can choose $[a, b] = [0, 1]$. With $S^1 \cong [0, 1]/(0 \sim 1)$ c therefore uniquely determines a continuous map in the space of loops $LM := C^0(S^1, M)$ (compact-open topology). Since a geodesic is not only continuous but also smooth, $c : [0, 1] \to M$ is in particular absolutely continuous. Furthermore $\int_0^1 g\big(\dot{c}(t), \dot{c}(t)\big)\mathrm{d}t$ is finite and thus c is a so-called H^1-curve. The set of H^1-loops on M is denoted by $H^1(S^1, M)$ or by ΛM. The subset $\Lambda M \subset LM$ has very useful properties:

- ΛM can be given the structure of smooth (infinite-dimensional) manifold.

- With its manifold topology the inclusion $\Lambda M \hookrightarrow LM$ is a homotopy equivalence.
- And, most important for us, there is a differentiable function $E_g : \Lambda \to \mathbb{R}$, called the energy function associated to a metric g on M, whose nonconstant critical points are exactly the closed geodesics of g.
  The function E_g moreover satisfies the so-called Palais-Smale condition, also called Condition C. This enables us to relate the topology of ΛM to the critical points of E_g.

The manifold ΛM is usually called the free loop space of M.

This is motivation enough to study the topology of ΛM. In the search for closed geodesics, this has already been successful a number of times. For instance, it is known that if M is compact and simply-connected and if the sequence of Betti numbers $\big(b_k(\Lambda M; F)\big)_k$, for some field F, is unbounded, then any Riemannian metric on M has infinitely many geometrically distinct closed geodesics ([**GM69**, Theorem 4]). This does not answer the question for spheres, since $H_i(\Lambda S^n; F)$ has at most one generator in each degree i if $n \geq 3$. For cases like the sphere one might still be able to extract information about closed geodesics from the homology of the free loop space by not only considering $H_*(\Lambda M)$ as a graded group, but also as a ring. Particularly useful in this regard are the Chas-Sullivan and the Goresky-Hingston product (see e.g. [**CS99**] and [**GH09**]). For example, in [**HR13**] predictions on the number of closed geodesics on odd-dimensional spheres under curvature pinching assumptions are made using the existence of (co-)homology classes that are nonnilpotent under these products.

The homology of ΛM with the Chas-Sullivan product is in fact the topic of this thesis. The idea roughly is to consider natural group actions on ΛM under which the energy E_g is invariant and which map a geodesic c to a geodesic c' which geometrically is the same geodesic. This means that we do not lose information about the geometrically distinct closed geodesics, but we might even gain information. The best examples for such a group actions are

(1) the shift of the starting point: If $c'(t) := c(t+s)$ then geometrically c and c' do not differ, but they are not equal as points in ΛM.
(2) the reversal of the orientation of a loop: $\bar{c}(t) := c(1-t)$ is the same geodesic but transversed in direction opposite to c.

It is the latter action that is treated in this work. Note that for non-reversible Finsler metrics c and $\bar{c}$ are not always considered to be geometrically equal.
If one looks at the quotient space $\Lambda M/\sim$ where c and c' are identified, one might get more information than by studying only ΛM; for example $\Lambda M/\sim$ might have more nontrivial homology. In addition, one might ask whether there is a multiplicative structure on $H_*(\Lambda M/\sim)$ that can be used. In [**Rad94**] for example, a class of S^1-equivariant cohomology of ΛM which is nonnilpotent under the cup product is used to estimate the number of closed geodesics.

In this thesis we study the quotient of ΛM by $\mathbb{Z}_2$-actions and define products on the quotient homology. More precisely, the content of this work is:

- In the first chapter we define ΛM and explain its importance for the search of closed geodesics in more detail. We also take a closer look at our guiding example (S^n, g_{st}): The function $E_{g_{st}} : \Lambda S^n \to M$ is in fact a Morse-Bott function and we can compute the homology $H_*(\Lambda S^n)$ from the critical set $Cr(E_{g_{st}})$ of $E_{g_{st}}$, which we know is equal to the constant curves and the great circles parametrized proportionally to arc length.

- In Chapter 2 we introduce a singular homology product on $H_*(\Lambda M)$, namely the so-called Chas-Sullivan product, which we denote by $*$:
$$* : H_i(\Lambda M) \times H_j(\Lambda M) \to H_{i+j-n}(\Lambda M).$$
Here $n = dim(M)$ and coefficients depend on whether M is orientable or not. The Chas-Sullivan product is defined using the concatenation of loops which start at a common point. It is thus in a way adapted to the closed geodesic problem. To illustrate what we mean, consider again the example (S^n, g_{st}). If $B_r \subset \Lambda S^n$ denotes the subspace of r-fold covered great circles, then one finds
$$H_i(\Lambda S^n) \cong H_i(S^n) \oplus \bigoplus_{r \in \mathbb{N}} H_{i-(2r-1)(n-1)}(B_r).$$
If c is a prime great circle, its r-th iterate c^r is an element of B_r. The concatenation $c^r \cdot c^l$ then equals c^{r+l} up to reparametriziation. This is also reflected in the fact that if $a \in H_i(\Lambda S^n)$ stems from a class of B_r and $b \in H_j(\Lambda S^n)$ from a class of B_l then $a * b \in H_{i+j-n}(\Lambda S^n)$ stems from a class of B_{r+l} ([**GH09**, Section 13]). Note that this is not true for every metric.
The map $ev_0 : \Lambda \to M,\ \gamma \mapsto \gamma(0)$ is a fibration whose fibre over $p \in M$ is the space of loops based at p, which is denoted by Ω_p. The Chas-Sullivan product relates to the Pontrjagin product defined on the homology of Ω_p and to the intersection product defined on $H_*(M)$. These relation are also shown in the second chapter since they will be made use of later in the text.
- In the third chapter we consider the actions of finite groups on ΛM. If G is finite and discrete and acts continuously on ΛM, then, since ΛM is topologically nice enough, there exist so called transfer homomorphism:
$$tr : H_i(\Lambda M/G) \to H_i(\Lambda M).$$
These transfers tr are very much like those associated to a covering map $p : \widetilde{X} \to X$, i.e. associating to a singular simplex $\sigma : \Delta^p \to X$ the sum $\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_{|G|}$ of the $|G|$ different lifts $\widetilde{\sigma}_i : \Delta^p \to \widetilde{X}$ of σ. In our case the quotient map $q : \Lambda M \to \Lambda M/G$ might not be a covering map. In fact, usually it is not, since the action of G is not free. Nevertheless, a transfer is defined and has similar properties as the covering transfer. It is then possible to define what we call the transfer product:
$$\begin{aligned} P_G : H_i(\Lambda M/G) \times H_j(\Lambda M/G) &\to H_{i+j-n}(\Lambda M/G) \\ (a,b) &\mapsto q_*\big(tr(a) * tr(b)\big) \end{aligned}$$
where $*$ is the Chas-Sullivan loop product of chapter 2.
We will consider $\mathbb{Z}_2$-actions. Our most important example is the action induced by the involution $\vartheta : \Lambda M \to \Lambda M,\ \vartheta(\gamma) := \bar{\gamma}$, where $\bar{\gamma}$ is γ but with opposite orientation as already mentioned above: $\bar{\gamma}(t) := \gamma(1-t)$. Its associated transfer product is denoted by P_ϑ and we have
$$ev_0 \circ \vartheta = ev_0,$$
$$\vartheta_*(a * b) = \vartheta_*(a) * \vartheta_*(b)$$
and so we prove for this action (and similar ones)

THEOREM. For a connected, compact, oriented manifold M the product P_ϑ is an associative, graded-commutative and unital product on the graded singular homology vector space $H_*(\Lambda M/\vartheta;\mathbb{Q})$. The map $(ev_0/\vartheta)_*$ is up to scaling an algebra homomorphism to the intersection algebra on $H_*(M;\mathbb{Q})$.

Here $\cdot/\vartheta$ stand for quotients under the action ϑ and induced maps on these quotients.
Another $\mathbb{Z}_2$-action we treat explicitly is closely related to the action above: The involution $\theta : \Lambda M \to \Lambda M$ is defined to be $\chi_{\frac{1}{2}} \circ \vartheta$, where $\chi_{\frac{1}{2}}$ shifts the starting point of a loop by $\frac{1}{2}$: $\chi_{\frac{1}{2}}(\gamma)(t) := \gamma(t+\frac{1}{2})$. We show

THEOREM. $\big(H_*(\Lambda M/\vartheta;\mathbb{Q}), P_\vartheta\big) \cong \big(H_*(\Lambda M/\theta;\mathbb{Q}), P_\theta\big)$ as $\mathbb{Q}$-algebras.

In the third chapter we also define products on $\Lambda M/\mathbb{Z}_2$ in a more direct way, namely by trying to concatenate equivalence classes of loops and hence imitating the construction of the Chas-Sullivan product. That is to say we try to define a product in the spirit of the following composition:

$$\begin{array}{c}
H_i(\Lambda M/\mathbb{Z}_2) \times H_j(\Lambda M/\mathbb{Z}_2) \\
\Big\downarrow{\scriptstyle \text{transfer}\times\text{transfer}} \\
H_i(\Lambda M) \times H_j(\Lambda M) \\
\Big\downarrow{\scriptstyle \times} \\
H_{i+j}(\Lambda M \times \Lambda M) \\
\Big\downarrow \\
H_{i+j}\big((\Lambda M \times \Lambda M)/\mathbb{Z}_2\big) \\
\Big\downarrow \\
H_{i+j}\big((\Lambda M \times \Lambda M)/\mathbb{Z}_2, (\Lambda M \times \Lambda M)/\mathbb{Z}_2 - \mathcal{F}/\mathbb{Z}_2\big) \\
\Big\downarrow{\scriptstyle \text{Thom isomorphism}} \\
H_{i+j-n}(\mathcal{F}/\mathbb{Z}_2) \\
\Big\downarrow{\scriptstyle \text{concatenation}} \\
H_{i+j-n}(\Lambda M/\mathbb{Z}_2)
\end{array}$$

where $\mathcal{F} \subset \Lambda M \times \Lambda M$ denotes the set of pairs of composable loops. $\mathcal{F}$ is a submanifold of finite codimension. We do this for the two actions generated by ϑ and θ respectively. It turns out that this is possible and the two products defined in the above way are denoted by A_ϑ and A_θ respectively. However, taking a closer look at the constructions it does not come as a big surprise that there is a relation between A_ϑ and P_ϑ and A_θ and P_θ respectively. In fact we show

THEOREM. On $H_*(\Lambda M/\theta;R)$ with M a connected, compact, oriented manifold and arbitrary coefficients R, we have that A_θ equals P_θ.

Surprisingly, $A_\vartheta = P_\vartheta$ does not hold in general. We show

THEOREM. Let M be an even-dimensional connected, compact, oriented manifold, then A_ϑ equals P_ϑ on rational homology.

In fact, A_ϑ is only defined for $\mathbb{Q}$-coefficients and it is only nontrivial for even-dimensional M. In other cases a variation of A_ϑ might be defined, but then a relation to P_ϑ or the Chas-Sullivan product is at least not obvious. It might be worth investigation this further, especially for coefficients other than $\mathbb{Q}$.

At the end of the chapter we also define a product B on equivariant homology $H_*(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2)$. It is denoted by B, standing for "bracket", since it is imitating the construction of the loop bracket on S^1-equivariant loop space homology defined in Section 17 of [**GH09**].

- In the last chapter we compute the products $A_\theta = P_\theta$, A_ϑ and P_ϑ on rational homology of quotients of the free loop space of spheres. We prove

  THEOREM. Let $n > 2$ and let ϑ be the orientation reversal of loops on ΛS^n. Then
  - for n odd, there exists a generator μ of $H_{3n-2}(\Lambda S^n/\vartheta; \mathbb{Q})$ which is not nilpotent in the algebra $(H_*(\Lambda S^n/\vartheta; \mathbb{Q}), P_\vartheta)$. Moreover, multiplication with μ, i.e.
  $$P_\vartheta(\cdot, \mu) : H_i(\Lambda S^n/\vartheta; \mathbb{Q}) \to H_{i+2n-2}(\Lambda S^n/\vartheta; \mathbb{Q})$$
  is an isomorphism for $i \geq 0$.
  - for n even, there exists a generator η of $H_{5n-4}(\Lambda S^n/\vartheta; \mathbb{Q})$ which is not nilpotent in the algebra $(H_*(\Lambda S^n/\vartheta; \mathbb{Q}), P_\vartheta)$. Moreover, multiplication with η, i.e.
  $$P_\vartheta(\cdot, \eta) : H_i(\Lambda S^n/\vartheta; \mathbb{Q}) \to H_{i+4n-4}(\Lambda S^n/\vartheta; \mathbb{Q})$$
  is an isomorphism for $i > 0$.

  And we prove a similar statement for θ.
  The existence of the nonnilpotent classes μ and η could be useful, even though they both come from powers of a single class of $H_*(\Lambda)$ which is nonnilpotent in the Chas-Sullivan ring.

  At the end of the fourth chapter we briefly consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action given by the combination of ϑ with θ. It turns out that for spheres and rational homology the quotient by this $\mathbb{Z}_2 \times \mathbb{Z}_2$-action does not carry more information than the quotients $\Lambda S^n/\vartheta$, $\Lambda S^n/\theta$. Also, an attempt of computing the bracket B constructed in Chapter 3 shows that computations get more involved when we take $\mathbb{Z}_2$ as coefficient ring.

Whether the above results could be used to find closed geodesics is not yet clear and would be part of further investigation. Application to other, homologically more complicated spaces than spheres or computation with coefficients other than $\mathbb{Q}$ should be considered.

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CHAPTER 1

The free loop space ΛM

(1) In the first section we define the space of continuous loops LM on a topological space M. We also show that there is a fibration $LM \to M$ which sends a loop to its starting point.
(2) In the second section M will be a smooth compact manifold and we equip the space of loops on M with a manifold structure. This manifold of loops is then called the free loop space of M and is denoted by ΛM. ΛM and LM are in fact homotopy equivalent.
(3) In the last section of this chapter we will make use of the manifold structure of ΛM to do Morse theory on it. The function we are interested in is the energy $E_g : \Lambda M \to \mathbb{R}$. $E = E_g$ depends on a chosen Riemannian metric g on M. The closed geodesics of the metric g are among the critical points of E and that is the entire reason why we are interested in the function E. For certain geometries, for example for the standard metric on S^n, the energy is a Morse-Bott function. It can then be used to compute the homology of ΛM.

1. Loop spaces

1.1. The space of loops. Let M be a topological space. We consider the space of continuous loops $LM := C^0(S^1, M)$ in M. LM carries the compact-open topology. The base point in S^1 in denoted by 0.

Proposition 1.1. *The evaluation map $ev_0 : LM \to M$, $\gamma \mapsto \gamma(0)$ is a Serre fibration. The fibre $ev_0^{-1}(p)$ over a point $p \in M$ is the space of loops based at p:*

$$\Omega_p := \{\gamma \in LM |\ \gamma(0) = p\} = ev_0^{-1}(p).$$

Proof, long version. We consider the space $PM := C^0(I, M)$ of continuous paths in M. Here $I = [0, 1]$ and PM is endowed with the compact-open topology.
It can be shown ([**LS15**, Satz 9.9] or [**Bre93**, Chapter VII, Theorem 6.13]) that the map $(ev_0, ev_1) : PM \to M \times M \cong C^0(\{0, 1\}, M)$, $\gamma \mapsto \big(\gamma(0), \gamma(1)\big)$ is a Serre fibration (even a Hurewicz fibration). (The evaluation ev_t at any time is continuous since $I = [0, 1]$ is (locally) compact. The inclusion of a subcomplex into a CW complex, e.g. $\{0, 1\} \hookrightarrow [0, 1]$, is a cofibration.)
Consider the diagonal embedding $M \overset{\Delta}{\hookrightarrow} M \times M$, along which we pull PM back:

$$\begin{tikzcd} M \times_M PM \arrow[r, hook, "pr_{PM}"] \arrow[d, "pr_M"'] & PM \arrow[d, "(ev_0, ev_1)"] \\ M \arrow[r, hook, "\Delta"'] & M \times M \end{tikzcd}$$

Here $M \times_M PM := \big\{(p,\gamma) \in M \times PM \mid (p,p) = \big(\gamma(0),\gamma(1)\big)\big\}$ is the so-called fibre product or pullback. It carries the subspace topology. The above commutative diagram is called a pullback diagram. The maps pr_M and pr_{PM} are the restrictions of the projections onto the factors of $M \times PM$ to the space subspace $M \times_M PM$. The map $pr_M : M \times_M PM \to M$ is then a Serre (even Hurewicz) fibration as well ([**Bre93**, Chapter VII, Theorem 6.14]).
Let now $q : I \to S^1$ be the quotient map identifying 0 with 1. It induces a continuous map $q^\sharp : LM \to PM$, $\gamma \mapsto \gamma \circ q$ ([**LS15**, Satz 4.19]). Since q is surjective, $q^\sharp$ is injective. In fact, $q^\sharp$ is an embedding: Let $\tilde{U} \subset LM$ be a basis element of the topology, i.e. $\tilde{U} = \bigcap_{i=1}^n M(K_i, V_i)$. As q is surjective $M(K_i,V_i) = M(q\big(q^{-1}(K_i)\big), V_i)$ and since q is proper as a continuous map from a compact to a Hausdorff space, $q^{-1}(K_i)$ is compact. Denoting the basis element $\bigcap_{i=1}^n M(q^{-1}(K_i), V_i)$ of the topology on PM by U, it follows that $\tilde{U} = \bigcap_{i=1}^n (q^\sharp)^{-1}\big(M(q^{-1}(K_i),V_i)\big) = (q^\sharp)^{-1}\big(\bigcap_{i=1}^n M(q^{-1}(K_i),V_i)\big) = (q^\sharp)^{-1}(U)$. Thus LM carries the topology induced by $q^\sharp$.
Consider the continuous map $ev_0 = ev : LM \to M$, $\gamma \mapsto \gamma(0)$. The diagram

$$\begin{tikzcd}
LM \arrow[dr, "h"] \arrow[drr, bend left, "q^*"] \arrow[ddr, bend right, "ev"'] & & \\
& M \times_M PM \arrow[r, hook] \arrow[d] & PM \arrow[d, "{(ev_0, ev_1)}"] \\
& M \arrow[r, hook, "\Delta"'] & M \times M
\end{tikzcd}$$

is then commutative. Here the map h is given by $h(\gamma) = \big(ev(\gamma), q^\sharp(\gamma)\big) = (\gamma(0), \gamma \circ q)$ and is clearly continuous. As $q^\sharp$ is an embedding, it is a homeomorphism onto its image. We have $q^\sharp(LM) = \{\gamma \in PM \mid \gamma(0) = \gamma(1)\}$ which equals $(ev_0, ev_1)^{-1}\big(\Delta(M)\big) = pr_{PM}(M \times_M PM)$. Let $f : q^\sharp(LM) \to LM$ be the continuous inverse of $q^\sharp$, then $id_M \times f : M \times_M PM \to LM$ is the continuous inverse for h. This holds since $M \times_M PM \subset M \times q^*(LM)$ and since product and subspace topology agree on $M \times q^*(LM)$. It follows that h is a homeomorphism and that $LM \xrightarrow{ev} M$ is a Serre (even Hurewicz) fibration.
The fibre over a point $p \in M$ obviously is $\Omega_p := \{\gamma \in LM \mid \gamma(0) = p\} = ev^{-1}(p)$. $\square$

PROOF, SHORT VERSION. Since the inclusion of the base point $i : 0 \hookrightarrow S^1$ is a cofibration and S^1 is (locally) compact and Hausdorff we can apply [**Bre93**, Chapter VII, Theorem 6.13] to deduce that $i^* = ev_0 : LM \to C^0(0, M) \cong M$ is a Hurewicz fibration. $\square$

If M is a topological manifold, then $ev_0 : LM \to M$ is in addition locally trivial. From now on we even take M to be a differentiable manifold.

1.2. The manifold of loops. From now on M will be a compact smooth Riemannian manifold. In this case, instead of investigating LM we can look at the Hilbert manifold ΛM of H^1-curves on M. Topologically this does not make much difference since LM and ΛM are homotopy equivalent.
In fact we wish to have continuous inclusions

$$C^\infty(S^1, M) \hookrightarrow \Lambda M \hookrightarrow LM$$

that are homotopy equivalences. This is in fact true by [**Kli78**, Theorem 1.2.10] or [**Moo17**, Theorem 1.5.1].
We can choose any Riemannian metric g on M and consider the induced distance d_g on M. The (compact-open) topology on $LM = C^0(S^1, M)$ is then induced by the distance d_∞

$$d_\infty(f, h) := \sup_{t \in S^1} d_g\big(f(t), h(t)\big).$$

This holds since S^1 is compact and Hausdorff and M is a metric space ([**LS15**, Satz 4.18]).

We start by defining Hilbert manifolds and the space ΛM of H^1-loops on M.

Definition 1.2. Let H be a separable Hilbert space.

- ([**Kli82**, Defintiton 1.1.1]) A topological Hilbert manifold modeled on H is a separable, metrizable space such that every point has a neighbourhood that is homeomorphic to H. (This is equivalent to being second countable, Hausdorff and locally homeomorphic to H.)
- ([**Kli82**, Defintiton 1.1.2]) A differentiable Hilbert manifold is a topological Hilbert manifold with a differentiable structure (defined by a differentiable ([**Kli82**, page 5]) atlas).

Definition 1.3. We define the set $H^1(S^1, M) := \{f \in LM | \forall t \in S^1 \ \exists$ charts with $t \in (U, \phi)$ and $f(t) \in (V, \psi)$ such that $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is of Sobolev class $H^1\}$ ([**Kli82**, Definition 2.3.1 (iii)]). By this we mean that $\phi(U) \xrightarrow{\psi \circ f \circ \phi^{-1}} \psi(V) \hookrightarrow \mathbb{R}^n$ is the unique continuous representative of an element of $H^1\big(\phi(U), \mathbb{R}^n\big) = \{g \in L^2\big((a, b) = \phi(U), \mathbb{R}^n\big) | \ \exists \partial g \in L^2\}$. Here ∂g denotes the weak derivative.
As done in [**Kli78**, page 8] this set of functions can also be described as the set of absolutely continuous maps $f : S^1(= [0, 1]/\{0, 1\}) \to M$ whose derivative f' (which is defined almost everywhere) is square-integrable, i.e. $\int_{S^1} g\big(f(t)\big)\big(f'(t), f'(f)\big)\mathrm{d}t < \infty$.
We set

$$\Lambda M := H^1(S^1, M)$$

We note that as sets we have $\Lambda M \subset LM$ and that for an $f \in \Lambda M$, the derivative f' is defined.
The set $\Lambda M = H^1(S^1, M)$ can be made into a differentiable Hilbert manifold:

- Choose a Riemannian metric g on M with exponential map exp and Levi-Civita connection ∇.
- Let $c \in C^\infty(S^1, M)$ be a smooth loop, so that $c^*(TM) \to S^1$ is a smooth bundle with pullback metric and pullback connection. By $H^1\big(c^*(TM)\big)$ we denote the set of H^1-sections of the bundle $c^*(TM)$, that is the completion of $C^\infty\big(c^*(TM)\big)$ with respect to the metric $g_1(X, Y) := \int_{S^1} g\big(X(t), Y(t)\big) + g\big(\nabla_{\dot{c}(t)}X(t), \nabla_{\dot{c}(t)}Y(t)\big)\mathrm{d}t$. We can think of $H^1(c^*(TM))$ as the set of continuous sections X for which $\nabla_{\dot{c}(t)}X(t)$ is defined almost everywhere and for which $g_1(X, X) < \infty$.
  For any smooth c, $H^1\big(c^*(TM)\big)$ is a separable Hilbert space of infinite dimension. Thus $H^1\big(c^*(TM)\big)$ and $H^1\big(d^*(TM)\big)$ are isometrically isomorphic for any two smooth loops c and d. This spaces are the model space for the Hilbert manifold ΛM.

- There is a open neighbourhood U of the zero section of $\pi : TM \to M$ such that $\pi \times \exp : U \to M \times M$ is a diffeomorphism onto an open neighbourhood of the diagonal $\Delta(M) \subset M \times M$. Since M is compact, we can choose U to be an open disk bundle. Let $H^1\big(c^*(U)\big)$ be the set of those sections X with $X(t) \in U \,\forall t \in S^1$ and define the map
$$\exp_c : H^1\big(c^*(U)\big) \to \Lambda M,\ X \mapsto \big(t \mapsto \exp \circ \tilde{c} \circ X(t)\big),$$
where $\tilde{c}$ is the bundle map in the pullback square
$$\begin{tikzcd} c^*(TM) \arrow[r, "\tilde{c}"] \arrow[d] & TM \arrow[d, "\pi"'] \\ S^1 \arrow[r, "c"'] & M. \end{tikzcd}$$
- One can show that $\left(\exp_c{}^{-1}, \exp_c\left(H^1\big(c^*(U)\big)\right)\right)_{c \in C^\infty(S^1,M)}$ defines a differentiable atlas for $\Lambda M = H^1(S^1, M)$.

Theorem 1.4. [**Kli78**, Theorem 1.2.9] *ΛM is a smooth Hilbert manifold of infinite dimension.*

H^1-vector fields are also defined along H^1-curves and the tangent bundle $T\Lambda M \to \Lambda M$ of ΛM has them as fibres: $H^1\big(f^*(TM)\big) \cong T_f\Lambda M$ is the fibre over $f \in \Lambda M$ ([**Kli78**, Section 1.3]).
Induced by the metric g on M, there is a metric g_1 on $H(S^1, M) = \Lambda M$: The metric
$$g_1(X, Y) := \int_{S^1} g(X(t), Y(t))\mathrm{d}t + \int_{S^1} g\big(\nabla_{\dot{c}}X(t), \nabla_{\dot{c}}Y(t)\big)\mathrm{d}t,$$
defined on $H^1\big(c^*(TM)\big)$ for smooth loops c, extends to all of $H^1(S^1, M)$ ([**Kli78**, Theorem 1.3.6]).

Theorem 1.5. [**Kli78**, Theorem 1.4.5] *or* [**Kli82**, Theorem 2.4.7] *If M is compact, $(\Lambda M, g_1)$ is a complete Riemannian manifold. That is, ΛM with the distance induced form g_1 is a complete metric space.*

Theorem 1.6. [**Kli82**, Lemma 2.4.6] *and* [**Kli78**, Theorem 1.2.10]

- *The inclusion*
$$\Lambda M \hookrightarrow LM$$
is continuous.
- *The inclusion*
$$\Lambda M \hookrightarrow LM$$
is a homotopy equivalence.

We close this section with

Proposition 1.7. *The evaluation map $ev_0 : \Lambda M \to M, \gamma \mapsto \gamma(0)$ is a locally trivial fibration. If M is connected, the fibres are all homeomorphic to Ω_p, the based loop space at any point $p \in M$. $\Omega_p := \{\gamma \in \Lambda M | \gamma(0) = p\} \subset \Lambda M$ is a smooth submanifold of ΛM of codimension n.*

A proof can be found in [**AS06**, Lemma 2.2].

From this proposition two properties follow which we are going to use later on:

(1) $\Omega_p \hookrightarrow \Lambda M \xrightarrow{ev_0} M$ has an associated long exact homotopy sequence of a fibration.
(2) Ω_p and its continuous counterpart $\{\gamma \in LM | \, \gamma(0) = p\}$ (for which we use the same name) are homotopy equivalent.

2. Morse theory on loop spaces

In order to do Morse theory we want (N, h) to be a smooth Riemannian Hilbert manifold and $f : N \to \mathbb{R}$ a smooth function, which satisfy (see e.g. [**PT88**, Chapter 9]):

(1) (N, h) is a complete Riemannian manifold. (We again mean metrically complete. This implies geodesic completeness but Hopf-Rinow does not necessarily hold in infinite dimensions, see e.g. Proposition 6.5 f. of Chapter VIII of [**Lan99**].)
(2) f is bounded from below, i.e. the exists a constant $C > -\infty$ such that $f \geq C$.
(3) f satisfies the Palais-Smale condition (sometimes also called "Condition C"), that is:
If $\{p_n\}$ is a sequence of points in N with
   - $|f(p_n)| \leq C$ for some $C < \infty$ and all $n \in \mathbb{N}$,
   - $||df(p_n)||_h \to 0$ as $n \to \infty$,

then there is a convergent subsequence $\{p_{n_k}\} \to p$.
We remark that it follows that p is a critical point of f.

On $H(S^1, M) = \Lambda M$ we define the energy function

$$E = E_g : \Lambda M \to \mathbb{R},\ c \mapsto \frac{1}{2} \int_{S^1} g\big(\dot{c}(t), \dot{c}(t)\big) \mathrm{d}t,$$

where g is the Riemannian metric on M we happen to care about.

In the case of the Riemannian manifold $(\Lambda M, g_1)$ with the function E_g defined on it, the above criteria are met and we can do Morse theory:

(1) At least for compact M, $(\Lambda M, g_1)$ is a complete Riemannian manifold: this is Theorem 1.5,
(2) The energy E is smooth ([**Kli78**, Theorem 1.3.9]) and clearly bounded from below: $E \geq 0$.
(3) At least for compact M, $E = E_g$ satisfies the Palais-Smale condition: This is Theorem 1.4.7 in [**Kli78**] or Theorem 2.4.9 in [**Kli82**]

We are thus ready to apply critical point theory to the function E_g on $\Lambda M = (\Lambda M, g_1)$ if M is compact. We have

- The critical points of E_g are the constant loops or the closed geodesics of the metric g ([**Kli78**, Theorem 1.3.11]).
- There is a self-adjoint (w.r.t. g_1) endomorphism $A_c : T_c\Lambda M \to T_c\Lambda M$ of the form "identity + compact operator" at critical points c of E such that

$$d^2E(c)(X, Y) = g_1(A_cX, Y)$$

([**Kli78**, Theorem 2.4.2]). It follows that there is a splitting

$$T_c\Lambda M = T_c^-\Lambda M \oplus T_c^0\Lambda M \oplus T_c^+\Lambda M$$

corresponding to the negative, zero and positive Eigenvalues of A_c. Moreover, the dimensions of the first two spaces are finite dimensional: $dim(T_c^-\Lambda M) < \infty$, $dim(T_c^0\Lambda M) < \infty$ ([**Kli78**, Corollary 1 of Theorem 2.4.2]).
The integer

$$\lambda_c := dim(T_c^-\Lambda M) \tag{2.1}$$

is called the index of the critical point c. We can thus assign a finite index to each closed geodesic.

Next, we ask whether for some metrics g, the function E_g does not only satisfies condition C, but is also a Morse function. In fact, this is the case for "generic" metrics:
By Morse function, we actually mean Morse-Bott function, since the critical points are never isolated in our case. We start with

DEFINITION 2.1. A connected submanifold B of ΛM is called a nondegenerate critical manifold for E_g if the following holds ([**Kli78**, page 58]):

- B consists entirely of critical points: $B \subset Cr(E_g)$, where $Cr(E_g) := \{\gamma \in \Lambda M |\ dE_g(\gamma) = 0\}$.
- The energy E_g is constant on B. In particular, $B = E_g^{-1}(k) \cap Cr(E_g) \subset \Lambda M$ is closed. (Here k is the energy of a $c \in B$.)
- The index $\lambda = \lambda_c$ is the same for all $c \in B$. We can then define $ind(B) := \lambda$.
- $T_c^0\Lambda = T_cB$ for all $c \in B$. In particular, the Hessian $d^2E_g(c)$ of E_g is nondegenerate in normal direction at every $c \in B$.
- B is invariant under shifting the starting point of loops, i.e. $c \in B \Rightarrow s.c \in B$ where for $s \in S^1$ we define $s.c(t) := c(t+s)\ \forall t \in S^1$.

Note that, due to Condition C, critical submanifolds are compact since the restriction of E_g to $Cr(E_g)$ is proper ([**Kli78**, Proposition 1.4.9] or [**PT88**, Lemma 9.1.1]).
Note also that the set of point curves $\Lambda M^0 := \{\gamma \in \Lambda M |\ E(\gamma) = 0\} \cong M$ in a nondegenerate critical submanifold with $ind(\Lambda M^0) = 0$ for any metric ([**Kli78**, Proposition 2.4.6]).

DEFINITION 2.2. The function $E = E_g$ is called Morse-Bott if $Cr(E_g)$ is the disjoint union of nondegenerate critical submanifolds.

Let k be an isolated critical value of E and assume that $B = E^{-1}(k)$ is a nondegenerate critical submanifold of index $ind(B) = \lambda$. Then, if a, b are regular values of E with $a < k < b$, if $[a, b]$ contains no other critical values and if $\Lambda M^{\leq v} := \{\gamma \in \Lambda M |\ E(\gamma) \leq v\}$ denotes the set of loops with energy less than or equal to $v \in \mathbb{R}$, we have

$$H_i(\Lambda M^{\leq b}, \Lambda M^{\leq a}; R) \cong H_{i-\lambda}(B; R)$$

with $R = \mathbb{Z}$ if the negative normal bundle $\Gamma^- \cong T^-\Lambda M_{|B} \to B$ of B is orientable and $R = \mathbb{Z}_2$ if the bundle is not orientable ([**Kli78**, Corollary 2.4.11]). Note that for any $c \in B$, $ind(B) = \lambda = dim(T_c^-\Lambda M) = rank(\Gamma^-)$ and the above is just the Thom isomorphism combined with the Morse Lemma ([**Kli78**, Chapter 2, Section 4] or concisely [**GH09**, Appendix D]).

In fact, if $B = E^{-1}(k) \cap Cr(E_g)$ is a nondegenerate critical submanifold, it follows from the Morse Lemma ([**Kli78**, Corollary 2.4.8]) that k is an isolated critical value. Hence, if E_g is Morse-Bott we get a strictly increasing filtration $\Lambda M^0 \subset \Lambda M^{\leq f_1} \subset \Lambda M^{\leq f_2} \subset \dots$ of ΛM with f_i a regular or a nondegenerate critical value for each $i \in \mathbb{N}$ and with only one critical value $k_i \in (f_{i-1}, f_i]$. With the spectral sequence associated to this filtration we might be able to compute the homology of the free loop space ΛM. This is for example the case when M is the round sphere: $(M, g) = (S^n, g_{st})$. In the case of the round sphere the spectral sequence degenerates on the first page. We will deal with (S^n, g_{st}) in the next section right below.

At the end of this section we like to mention that indeed for "most" metrics E_g is Morse-Bott. This is the content of the so-called bumpy metric theorem. It says that "metrics with no symmetries are generic and generic metrics are Morse-Bott". This is proved in [**Ano82**] or in [**Moo17**, Theorem 2.7.1].

3. Loop space homology of spheres

As mentioned in the last section, the homology of ΛS^n can be determined using Morse theory. If g_{st} denotes the standard metric on S^n, then $E_{g_{st}}$ is a perfect Morse-Bott function for any coefficients ([**Zil77**, Theorem 5 + Theorem 8] or [**GH09**, Theorem 13.4 + Proposition 13.2]). We have that the critical set $Cr(E_{g_{st}}) = \{\text{point curves}\} \sqcup \bigsqcup_{r\in\mathbb{N}} B_r$ where B_r is the set of r-fold covered great circles with each great circle parametrized proportionally to arc length. Each B_r a is connected $(2n-1)$-dimensional manifold diffeomorphic to the unit tangent bundle T^1S^n of S^n and a nondegenerate critical manifold for $E_{g_{st}}$. $E_{g_{st}}(B_r) = 2\pi^2 r^2$ and hence for the singular homology groups we have

$$H_i(\Lambda S^n;\mathbb{Z}) = H_i(S^n;\mathbb{Z}) \oplus \bigoplus_{r\in\mathbb{N}} H_i\big(\Lambda S^{n \leq 2\pi^2 r^2}, \Lambda S^{n < 2\pi^2 r^2};\mathbb{Z}\big) = H_i(S^n;\mathbb{Z}) \oplus \bigoplus_{r\in\mathbb{N}} H_{i-\lambda_r}(T^1S^n;\mathbb{Z}).$$

Here $ind(B_r) =: \lambda_r = (2r-1)(n-1)$ is the Morse index of each point (i.e. closed geodesic) in B_r ([**Kli82**, Example 2.5.7 (iib)]).

Let us take a closer look at the manifolds B_r:
Given any metric g on the compact manifold M, the isometry group $Iso(g)$ acts smoothly on M ([**KN63**, Theorem 3.4]). Hence the associated map $\rho : Iso(g) \to Diffeo(M) \subset Homeo(M) \subset C^0(M, M)$ is continuous.

- Since S^1 and M are locally compact, $Iso(g)$ acts continuously on ΛM ([**LS15**, Folgerung 4.23]):
$$Iso(g) \times \Lambda M \to \Lambda M,\ (h, \gamma) \mapsto \rho(h) \circ \gamma$$
is continuous.
- E_g is invariant under this action:
$$\begin{aligned} E_g(\rho(h) \circ \gamma) &= \frac{1}{2}\int_{S^1} g\big(d\rho(h)(\gamma(t))\dot\gamma(t), d\rho(h)(\gamma(t))\dot\gamma(t)\big)\mathrm{d}t \\ &= \frac{1}{2}\int_{S^1} g\big(\dot\gamma(t), \dot\gamma(t)\big)\mathrm{d}t \\ &= E_g(\gamma). \end{aligned}$$

Now let $(M, g) = (S^n, g_{st})$. Then $Iso(S^n, g_{st}) = O(n+1)$ and if γ_r is a r-fold covered great circle, then its orbit $O(n+1)\cdot\gamma_r = B_r$. If $\gamma_r = c_{p,rv}$ is the geodesic uniquely determined by the point $p \in S^n$ and the unit vector $v \in T_pS^n$, then γ_r is fixed under rotations in the orthogonal complement in $S^n \subset \mathbb{R}^{n+1}$ of the span of p, v, thus the stabilizer $O(n+1)_{\gamma_r}$ of γ_r is isomorphic to $O(n-1)$. It follows that the map

$$\nu_r : O(n+1)/O(n-1) \longrightarrow \Lambda S^n;\ [h] \mapsto \rho(h) \circ \gamma_r,$$

is a continuous closed $O(n+1)$-equivariant embedding with image B_r.

Proof for the continuous case. The map

$$O(n+1) \times \{\gamma_r\} \cong O(n+1) \longrightarrow \Lambda S^n,\ h \mapsto \rho(h) \circ \gamma_r,$$

is a continuous map and induces a continuous injective map

$$\nu_r : O(n+1)/O(n-1) \longrightarrow \Lambda S^n;$$

with image B_r. Since $O(n+1)/O(n-1)$ is compact and Hausdorff ([**Bre72**, Chapter 1, Theorem 3.1]), $\nu_r : O(n+1)/O(n-1) \longrightarrow B_r$ is a homeomorphism and since $B_r = Cr(E) \cap E^{-1}(2\pi^2 r^2)$ is compact by the Palais-Smale condition, $\nu_r : O(n+1)/O(n-1) \hookrightarrow \Lambda S^n$ is a closed embedding. In particular, B_r is a (smooth) submanifold of dimension $\frac{n(n+1)}{2} - \frac{(n-1)(n-2)}{2} = 2n-1$ since $O(n+1)/O(n-1)$ is.
For $a \in O(n+1)$ we have $\nu_r(a \cdot [h]) = \nu_r([ah]) = \rho(a) \circ \rho(h) \circ \gamma_r = \rho(a) \circ \nu_r([h]) = a \cdot \nu_r([h])$. $\square$

More generally, this is true if $Iso(g)$ is compact, which holds for any compact Riemannian manifold (M,g) ([**KN63**, Theorem 3.4]). The stabilizer $Iso(M,g)_\gamma$ is closed (hence also compact) since it is the preimage of $\{\gamma\}$ under the map $Iso(M,g) \to \Lambda M$, $h \mapsto h \cdot \gamma := \rho(h) \circ \gamma$, thus $Iso(g)/Iso(M,g)_\gamma$ is compact Hausdorff and $Iso(M,g)/Iso(M,g)_\gamma \to \Lambda M$, $[h] \mapsto h \cdot \gamma$ a closed embedding.

Back to spheres: For the standard embedding $S^n \subset \mathbb{R}^{n+1}$ we have that $TS^n = \{(p,v) \in S^n \times \mathbb{R}^{n+1} |\ p \cdot v = 0\}$ (topologized as a subspace of $S^n \times \mathbb{R}^{n+1}$) and hence $T^1S^n = \{(p,v) \in S^n \times \mathbb{R}^{n+1} |\ p \cdot v = 0 \text{ and } ||v|| = 1\} = \{(p,v) \in S^n \times S^n |\ p \cdot v = 0\} =: V_2(\mathbb{R}^{n+1})$ is the Stiefel manifold $V_2(\mathbb{R}^{n+1})$ of orthonormal two-frames in $\mathbb{R}^{n+1}$.
$O(n+1)$ acts continuously on $V_2(\mathbb{R}^{n+1})$ via $\big(h,(p,v)\big) \mapsto \big(h(p),h(v)\big)$. Obviously $V_2(\mathbb{R}^{n+1}) = O(n+1) \cdot (e_1,e_2)$ where (e_1,e_2) is the standard basis of $\mathbb{R}^2 \subset \mathbb{R}^{n+1}$. Since $O(n-1) \cong \{id_{\mathbb{R}^2}\} \times O(n-1) \subset O(n+1)$ is the stabilizer of (e_1,e_2) we have

$$\begin{tikzcd} T^1S^n = V_2(\mathbb{R}^{n+1}) \arrow[rrr, bend left, "f_r"] & O(n+1)/O(n-1) \arrow[l, "\cong"'] \arrow[r, "\nu_r", "\cong"'] & B_r \arrow[r, hook] & \Lambda S^n. \end{tikzcd}$$

Let now $\gamma : S^1 \to S^n$ be the simple closed geodesic with initial conditions p, v, i.e. γ is injective on $S^1 \setminus \{0\}$ where $0 \in S^1$ denotes the base point. γ is a prime closed geodesic. If $\gamma_r = \gamma^r$ denotes the r-fold iterate of γ, i.e. $\gamma_r(t) = \gamma(rt)$, then γ_r has (p, rv) as initial data. We also write that as $\gamma = c_{p,v}$ and $\gamma_r = c_{p,rv}$. Via the standard embedding $S^n \subset \mathbb{R}^{n+1}$ we have $\rho(h) \circ \gamma_r = h \cdot \gamma_r = h \cdot c_{p,rv} = c_{h(p),h(rv)} = c_{h(p),rh(v)}$. Hence, if we also use $(e_1,e_2) \in T^1S^n$ to define ν_r we get

$$f_r : T^1S^n \to \Lambda S^n,\ (p,v) \mapsto c_{p,rv}.$$

This is a continuous closed $O(n+1)$-equivariant embedding.

The above is more generally true for Riemannian manifolds all of whose geodesics are closed and simply periodic with common prime length l, i.e. if γ is a prime closed geodesic then $\gamma(0) = \gamma(1)$, $\dot{\gamma}(0) = \dot{\gamma}(1)$, γ injective on $(0,1)$ and $L(\gamma_{|[0,1]}) = l$. In this case, the (non-constant) critical set is the disjoint union of nondegenerate compact submanifolds consisting of iterated prime geodesics: $Cr(E) = \{\text{point curves}\} \sqcup \bigsqcup_r \Sigma_r$, Σ_r is diffeomorphic to T^1M for each r ([**GH09**, Section 13]).

CHAPTER 2

The Chas-Sullivan product

In this chapter we do the following:

(1) In the first section we define the Chas-Sullivan product. It is a product on the homology of the free loop space ΛM of a compact manifold M. It is defined using the concatenation of composable loops and it is sometimes also called the loop product.

(2) In the second section we construct explicit tubular neighbourhoods of the space $\mathcal{F} \subset \Lambda M \times \Lambda M$ of composable loops.

(3) In section three we relate the Chas-Sullivan product on ΛM to other products which are defined on M itself and on the based loops ΩM respectively. The relations come from the fibration $\Omega M \to \Lambda M \xrightarrow{ev_0} M$.

1. Definition of the Chas-Sullivan product

Let M now be a fixed finite-dimensional compact and connected manifold. Let $\Lambda := \Lambda M$ denote the Hilbert manifold of closed curves on M. Let M have dimension n.
We call $\mathcal{F} := \{(\alpha, \beta) \in \Lambda \times \Lambda |\, \alpha(0) = \beta(0)\}$ the figure-eight space. $\mathcal{F}$ fits into the follwing pullback diagram:

(1.1)
$$\begin{tikzcd}
\mathcal{F} \arrow[dr, "f"', "\cong"] \arrow[drr, hook, bend left, "i_{\mathcal{F}}"] \arrow[ddr, bend right, "ev_0"'] & & \\
& M \times_{M^2} \Lambda^2 \arrow[r, hook, "pr_{\Lambda^2}"] \arrow[d] & \Lambda \times \Lambda \arrow[d, "ev_0 \times ev_0"] \\
& M \arrow[r, hook, "\Delta"'] & M \times M,
\end{tikzcd}$$

where $ev_0 : \mathcal{F} \to M$ is $ev_0(\alpha, \beta) := \alpha(0) = \beta(0)$, $i_{\mathcal{F}} : \mathcal{F} \to \Lambda \times \Lambda = \Lambda^2$ is the inclusion and $\Delta : M \hookrightarrow M \times M = M^2$, $p \mapsto (p,p)$ is the diagonal embedding. Here $M \times_{M^2} \Lambda^2$ denotes the fibre-product or pullback and it is the subspace of $M \times \Lambda \times \Lambda$ defined by $M \times_{M^2} \Lambda^2 := \{(p, \alpha, \beta) \in M \times \Lambda \times \Lambda |\, \Delta(p) = (ev_0 \times ev_0)(\alpha, \beta)\}$.
(ev_0 is continuous since it can be written as $pr \circ ev_0 \times ev_0 \circ i_{\mathcal{F}}$ where $pr : M \times M \to M$ is the projection to either factor. Since $ev_0 \times ev_0 \circ i_{\mathcal{F}}$ maps into $\Delta(M)$, the outer compositions in the above diagram are certainly equal.)
The homeomorphism f explicitly is $f(\alpha, \beta) = \big(ev_0(\alpha, \beta), i_{\mathcal{F}}(\alpha, \beta)\big) = \big(\alpha(0), (\alpha, \beta)\big)$. It is the unique continuous map that makes the diagram commutative. The composition $M \times_{M^2} \Lambda^2 \hookrightarrow M \times \Lambda \times \Lambda \to \mathcal{F} \subset \Lambda \times \Lambda$ provides a continuous inverse, it is $f^{-1}\big(p, (\alpha, \beta)\big) = (\alpha, \beta)$. Clearly, $pr_{\Lambda^2} = i_{\mathcal{F}} \circ f^{-1}$ is an embedding. It is also a closed embedding since $\mathcal{F} = (ev_0 \times ev_0)^{-1}\big(\Delta(M)\big)$ and M is closed in $M \times M$.

The Chas-Sullivan product

$$* : H_i(\Lambda) \times H_j(\Lambda) \to H_{i+j-n}(\Lambda)$$

is defined by (see [**GH09**, Section 5])

$$\begin{tikzcd}
H_i(\Lambda) \times H_j(\Lambda) \arrow[d, "(-1)^{n(n-j)}\times"'] \\
H_{i+j}(\Lambda \times \Lambda) \arrow[d] \arrow[dd, bend left, "i_{\mathcal{F}!}"] \\
H_{i+j}(\Lambda \times \Lambda, \Lambda \times \Lambda - \mathcal{F}) \arrow[d, "\text{Thom isomorphism}"'] \\
H_{i+j-n}(\mathcal{F}) \arrow[d, "\phi_*"'] \\
H_{i+j-n}(\Lambda).
\end{tikzcd}$$

Here $H_i(\cdot)$ denotes singular homology. Which coefficients we use will be clear from the context. The Chas-Sullivan product was originally defined in [**CS99**] with a different definition.

We explain the different maps in this definition:

- The first map is just the homological cross product $\times$ together with the sign-correction $(-1)^{n(n-j)}$ for nicer algebraic properties as we will see later.

- The map $i_{\mathcal{F}!}$ is a so-called "Gysin" or "umkehr" map. It is defined to be the composition of the map $H_{i+j}(\Lambda \times \Lambda) \to H_{i+j}(\Lambda \times \Lambda, \Lambda \times \Lambda - \mathcal{F})$ induced by inclusion and the Thom isomorphism. That the latter exists follows from the fact that $\mathcal{F}$ is a Hilbert submanifold of $\Lambda \times \Lambda$: One can show that $ev_0 \times ev_0 : \Lambda \times \Lambda \to M \times M$ is a submersion. Thus $\mathcal{F} = (ev_0 \times ev_0)^{-1}\big(\Delta(M)\big)$ is a submanifold of codimension n. It follows that there exists, up to isotopy, a unique tubular neighbourhood $U_{\mathcal{F}}$ of $\mathcal{F}$ in $\Lambda^2 := \Lambda \times \Lambda$ ([**Lan99**, Chapter IV, Sections 5 and 6 and Chapter VII, Section 4]). This means that there is a vector bundle $N_{\mathcal{F}} \to \mathcal{F}$ of rank n and an open embedding $t_{\mathcal{F}} : N_{\mathcal{F}} \hookrightarrow \Lambda^2$ with image $U_{\mathcal{F}}$, such that the diagram

$$\begin{tikzcd}
N_{\mathcal{F}} \arrow[r, hook, "t_{\mathcal{F}}"] & \Lambda^2 \\
\mathcal{F} \arrow[u, hook, "z"] \arrow[ur, hook, "i_{\mathcal{F}}"'] &
\end{tikzcd}$$

  commutes. Here z is the zero section of the bundle. Since $\mathcal{F}$ is closed in $\Lambda^2 = (\Lambda^2 - \mathcal{F}) \cup U_{\mathcal{F}}$ we have, by excision, that

$$H_{i+j}(\Lambda^2, \Lambda^2 - \mathcal{F}) \cong H_{i+j}(U_{\mathcal{F}}, U_{\mathcal{F}} - \mathcal{F}) \cong H_{i+j}(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}).$$

We can then cap with a (chosen) Thom class $\tau_{\mathcal{F}} \in H^n(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F})$ of $N_{\mathcal{F}}$, which is an isomorphism by the Thom isomorphism theorem ([**GH09**, Appendix B]) :

$$H_{i+j}(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}) \xrightarrow[\cong]{\cap \tau_{\mathcal{F}}} H_{i+j-n}(N_{\mathcal{F}}) \cong H_{i+j-n}(\mathcal{F}).$$

Clearly, we have to take appropriate coefficients for homology here: If M is orientable, then so is $N_{\mathcal{F}}$ (see below) and we can take any coefficient ring. Otherwise we have to take $\mathbb{Z}_2$ as coefficient domain.

- On $\mathcal{F}$ the composition of loops can be defined ([**GH09**, Section 2]): The "concatenation of loops" $\phi = \phi_{\frac{1}{2}} : \mathcal{F} \to \Lambda$ is a continuous map defined by

$$\phi(\gamma, \delta)(t) = \phi_{\frac{1}{2}}(\gamma, \delta)(t) := \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \delta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.$$

  We sometimes also write $\gamma \cdot \delta$ for $\phi(\gamma, \delta)$.

We now list some properties of the Chas-Sullivan product: Let $*'$ denote the Chas-Sullivan product without sign-correction, that is:

$$a *' b = \phi_*\big((a \times b) \cap \tau_{\mathcal{F}}\big),$$

where we omit the map $H_*(\Lambda M^2) \to H_*(\Lambda M^2, \Lambda M^2 - \mathcal{F})$, the excision and tubular neighbourhood map from the notation.

The Chas-Sullivan product is

- *associative* ([**CS99**, Theorem 3.3] or [**HW18**, Theorem 2.5]):
  The map $H : \{(\alpha, \beta, \gamma) \in \Lambda \times \Lambda \times \Lambda |\, \alpha(0) = \beta(0) = \gamma(0)\} \times [0,1] \to \Lambda$ defined by

$$H(\alpha, \beta, \gamma, s)(t) = \begin{cases} \alpha(\frac{4t}{1+s}), & 0 \le t \le \frac{1+s}{4}, \\ \beta(4t - 1 - s), & \frac{1+s}{4} \le t \le \frac{2+s}{4}, \\ \gamma(\frac{4t-2-s}{2-s}), & \frac{2+s}{4} \le t \le 1, \end{cases}$$

  is a homotopy between $\phi \circ (\phi \times id)$ and $\phi \circ (id \times \phi)$. Note that since this homotopy leaves the base point fixed, it can also be used to show the associativity of the Pontrjagin product on Ω. In fact, associativy of the Pontrjagin product

$$H_i(\Omega) \times H_j(\Omega) \to H_{i+j}(\Omega),\ a \star b := \phi_*(a \times b)$$

  follows immediately form the above homotopy. To prove the associativity of the Chas-Sullivan product with our definition directly furhter inspections are necessary. A first step in this direction is to see that the intersection product on M is associative and given, up to sign, by capping with the Thom class of $\Delta : M \hookrightarrow M \times M$. The Chas-Sullivan product relates to the intersection product on M via $ev_0 : \Lambda M \to M$ (see Section 3).

- *graded commutative* ([**GH09**, Proposition 5.2]):
Let $\mathrm{T} : \mathcal{F} \to \mathcal{F},\ (\alpha, \beta) \mapsto (\beta, \alpha)$. Denoting $\phi(\alpha, \beta)$ by $\alpha \cdot \beta$, we have
$$\begin{tikzcd}
(\alpha,\beta) \arrow[r, mapsto, "\phi"'] \arrow[d, mapsto, "\mathrm{T}"'] & \alpha\cdot\beta \arrow[d, mapsto, "G"', "\simeq"] \\
(\beta,\alpha) \arrow[r, mapsto, "\phi"'] & \beta\cdot\alpha.
\end{tikzcd}$$
i.e.
$$\phi_* = \phi_* \circ \mathrm{T}_*$$
via the homotopy $G : \mathcal{F} \times [0, 1] \to \Lambda$ given by
$$G(\alpha, \beta, s)(t) = \alpha \cdot \beta(t + \tfrac{s}{2}) = \begin{cases} \alpha(2t + s), & 0 \leq t + \frac{s}{2} \leq \frac{1}{2} \\ \beta(2t + s - 1), & \frac{1}{2} \leq t + \frac{s}{2} \leq 1 \end{cases}.$$
with $G_0 = \phi$ and $G_1 = \phi \circ \mathrm{T}$.
Note that the homotopy G does not leave the base point fixed and it can thus not be used for the Pontrjagin product, which indeed is not commutative in general.
It follows that
$$\begin{aligned} b *' a &= \phi_*\big((b \times a) \cap \tau_{\mathcal{F}}\big) \\ &= (-1)^{|a||b|}\phi_*\big(\mathrm{T}_*(b \times a) \cap \tau_{\mathcal{F}}\big) \\ &= (-1)^{|a||b|}\phi_* \circ \mathrm{T}_*\big((b \times a) \cap \mathrm{T}^*(\tau_{\mathcal{F}})\big) \\ &= (-1)^{|a||b|+n}\mathrm{T}_* \circ \phi_*\big((b \times a) \cap \tau_{\mathcal{F}})\big) \\ &= (-1)^{|a||b|+n} a *' b, \end{aligned}$$
where $|a|$ is the degree of a, i.e. $a \in H_{|a|}(\Lambda M)$.
- *unital* if M is a compact oriented manifold ([**HW18**, Theorem 2.5]):
The inclusion of point curves $c : M \hookrightarrow \Lambda M^0 \subset \Lambda M,\ p \mapsto (t \mapsto p\ \forall t)$ is injective in homology since it is a section of the evaluation $ev_0 \circ c = id_M$. The image $E := c_*([M])$ of the orientation class is a two-sided unit element for the Chas-Sullivan product .

2. Explicit tubes around the figure-eight space

We start with showing that $N_{\mathcal{F}}$, the normal bundle of $\mathcal{F}$ in Λ^2, is the pullback of the normal bundle N_M of $\Delta(M) \cong M$ in $M^2 = M \times M$:

Lemma 2.1. *$N_{\mathcal{F}} \cong ev_0{}^*(N_M)$.*

Proof. This holds since $ev_0 \times ev_0 : \Lambda^2 \to M^2$ is a submersion (compare e.g. [**Kli82**, Proposition 2.4.1]). Therefore $(ev_0 \times ev_0)^{-1}\big(\Delta(M)\big) = \mathcal{F}$ is a submanifold of codimension n. Also due to transversality, the composition
$$T\Lambda^2|_{\mathcal{F}} \xrightarrow{d(ev_0 \times ev_0)|_{\mathcal{F}}} TM^2|_{\Delta(M)} \longrightarrow TM^2|_{\Delta(M)}/T\Delta(M)$$
is fibrewise surjective. Since $T\mathcal{F}$ is in the kernel of this bundle map we get a map
$$f : N_{\mathcal{F}} := \frac{T\Lambda^2|_{\mathcal{F}}}{T\mathcal{F}} \to \frac{TM^2|_M}{TM} =: N_M$$

which is a surjective bundle map. For dimensional reasons f is fibrewise an isomorphism. We therefore have a commutative diagram

$$\begin{tikzcd}
N_{\mathcal{F}} \arrow[dr, "\cong", "g"'] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& {ev_0}^*(N_M) \arrow[r, "pr_{N_M}"] \arrow[d] & N_M \arrow[d] \\
& \mathcal{F} \arrow[r, "ev_0"] & M
\end{tikzcd}$$

which shows that g is fibrewise an isomorphism and thus it is a bundle isomorphism over $\mathcal{F}$. That is, the normal bundle of $\mathcal{F}$ is (isomorphic to) the pullback of the normal bundle of $\Delta(M) \cong M$. □

Let $t_{\mathcal{F}} : N_{\mathcal{F}} \cong {ev_0}^*(N_M) \hookrightarrow \Lambda^2$ be a tubular neighbourhood embedding with image an open neighbourhood $U_{\mathcal{F}}$ of $\mathcal{F} \subset \Lambda^2$. Also, let us denote by $t_M : N_M \hookrightarrow M \times M$ a tubular neighbourhood embedding of M into $M \times M$ and by $U_M := t_M(N_M)$ its image. We can ask whether this can be lifted, i.e. $U_{\mathcal{F}} = (ev_0 \times ev_0)^{-1}(U_M)$. It can; in [**HW18**] they give a commutative diagram like

$$\begin{tikzcd}
{ev_0}^*(N_M) \arrow[r, hook, "t_{\mathcal{F}}"] \arrow[d, "pr_{N_M}"'] & \Lambda \times \Lambda \arrow[d, "ev_0 \times ev_0"] \\
N_M \arrow[r, hook, "t_M"] & M \times M.
\end{tikzcd} \tag{2.1}$$

Since we are going to use them in this and the next chapter, we will now describe these tubular neighbourhood maps $t_{\mathcal{F}}$ and t_M concretely:

LEMMA 2.2. *Let (M, g) be a compact Riemannian manifold. Then its tangent bundle TM and the normal bundle N_M of $\Delta M \subset M \times M$ are isomorphic. In particular, TM is orientable if and only if N_M is.*
Moreover, since M is compact, there exists an $\varepsilon > 0$ such that the maps

$$s'_M : D_\varepsilon TM \hookrightarrow M \times M,\ (p, v) \mapsto \big(\exp(p, -v), \exp(p, v)\big)$$
$$t'_M : D_\varepsilon TM \hookrightarrow M \times M,\ (p, v) \mapsto \big(p, \exp(p, v)\big)$$

embed the disk bundle $D_\varepsilon TM := \{(p, v) \in TM |\ |v|_g < \varepsilon\}$ as tubular neighbourhoods into M^2. In particular, they both restrict to the diagonal inclusion $\Delta : M \hookrightarrow M^2$ on the zero section. Here exp *is the Riemannian exponential map associated to g.*

PROOF. We first look at the normal N_M bundle of M inside $M \times M$, i.e. the quotient bundle $N_M := \Delta^*\big(T(M \times M)\big)/TM \to M$.
We begin with a few simple observations: Let $i_{\Delta(M)} : \Delta(M) \hookrightarrow M^2$ be the inclusion of the diagonal. Then

$$T(M \times M)|_{\Delta(M)} \cong i^*_{\Delta(M)}\big(T(M \times M)\big)$$

canonically ([**Kna13**, Lemma 1.2.5]). Let $pr_i : M \times M \to M,\ i = 1, 2$ be the projection onto one of the two factors. Since $\Delta : M \to \Delta(M)$ is continuous with continuous inverse

$pr_{i|\Delta(M)} = pr_i \circ i_{\Delta(M)} : \Delta(M) \to M$, we have

$$\Delta^*\big(T(M\times M)\big) = (i_{\Delta(M)}\circ\Delta)^*\big(T(M\times M)\big) \cong i^*_{\Delta(M)}\big(T(M\times M)\big).$$

The three bundles

$$\begin{tikzcd}
\Delta^*\big(T(M\times M)\big) \arrow[r, "\cong"'] \arrow[d] & i^*_{\Delta(M)}\big(T(M\times M)\big) \arrow[r, "\cong"'] \arrow[d] & T(M\times M)|_{\Delta(M)} \arrow[d] \\
M \arrow[r, "\Delta"'] & \Delta(M) \arrow[r, "="'] & \Delta(M)
\end{tikzcd}$$

as thus all isomorphic via $\big(p,(p,p),(v,w)\big) \mapsto \big((p,p),(p,p),(v,w)\big) \mapsto \big((p,p),(v,w)\big)$.
The injective bundle map $TM \to T(M^2)$ given by $(p,v) \mapsto (p,p,v,v)$ embeds TM as a subbundle into $T(M\times M)$. Its image obviously is the set $T\Delta(M)$ of vectors tangent to $\Delta(M)$. (This map actually is the differential $d\Delta : TM \to T(M\times M)$ of the smooth embedding Δ.) Hence either side of the isomorphism

$$\begin{tikzcd}
\Delta^*\big(T(M\times M)\big)/TM \arrow[r, "\cong"'] \arrow[d] & T(M\times M)|_{\Delta(M)}/T\Delta(M) \arrow[d] \\
M \arrow[r, "\cong", "\Delta"'] & \Delta(M)
\end{tikzcd}$$

can be viewed as the normal bundle N_M of $\Delta(M) \subset M\times M$.
For the construction of tubular neighbourhoods we rather like to view the bundles TM and N_M as subbundles of $T(M\times M)|_{\Delta(M)}$: We have the exact sequence

$$0 \longrightarrow T\Delta(M) \longrightarrow T(M\times M)|_{\Delta(M)} \longrightarrow N_M \cong T(M\times M)|_{\Delta(M)}/T\Delta(M) \longrightarrow 0.$$

Let us equip $M\times M$ with a Riemannian metric, then $N_M \cong T(M\times M)|_{\Delta(M)}/T\Delta(M) \cong T\Delta(TM)^\perp$. The simplest metric is the product metric, that is on $T(M\times M) \cong TM\times TM$ we define the metric G as "twice" the metric on TM: $G\big((v,w),(v',w')\big) = g(v,v') + g(w,w')$. Then

$$TM \cong T\Delta(M) = \{(p,p,v,w) \in T(M\times M)_{|\Delta(M)} |\ v = w\},$$
$$N_M \cong T\Delta(M)^\perp = \{(p,p,v,w) \in T(M\times M)_{|\Delta(M)} |\ v = -w\}.$$

Note that using this, we also get a bundle isomorphism $TM \cong N_M$ via $(p,v) \mapsto \big((p,p),(-v,v)\big)$. Thus if M is orientable then so are $N_M \cong TM$ and $N_{\mathcal{F}} \cong ev_0{}^*(N_M)$ ([**MS74**, Lemma 11.5]). We have $(TM\times TM)_{|\Delta(M)} \cong TM \oplus N_M$ via

$$(p,p,v,w) = \big(p,p,\tfrac{1}{2}(v+w),\tfrac{1}{2}(v+w)\big) + \big(p,p,\tfrac{1}{2}(v-w),\tfrac{1}{2}(w-v)\big)$$

where N_M is embedded as $(p,v) = \big((p,p),(-v,v)\big)$ as above.

Consider an (open) ε-disk bundle $D_\varepsilon N_M := \{(p,v)\in N_M |\ |v| < \varepsilon\} \subset N_M$. Here the metric is induced by an embedding into $T(M^2)$. We choose ε small enough such that Riemannian exponential map associated to a chosen metric on $T(M^2)$ is defined and smooth on $D_\varepsilon N_M$ and maps $D_\varepsilon N_M$ diffeomorphically onto a tubular neighbourhood of $\Delta(M)$ in M^2. Such an ε exists for any metric by the usual proof of the tubular neighbourhood theorem. Note that we can take ε to be a constant throughout $\Delta(M)$ since M is compact.
We choose the product metric as above and use it to identify N_M with $TM^\perp$. We also use its

exponential map $\mathrm{Exp}: T(M^2) \to M^2$ which is given by $\mathrm{Exp}\big((p,q,v,w)\big) = \big(\exp(p,v), \exp(q,w)\big)$ where exp is the exponential map of M. Note that the domain of Exp is all of $T(M^2)$ since $M \times M$ is compact. With ε small enough and

$$D_{\sqrt{2}\varepsilon} N_M := \{(p,p,-v,v) \in T(M^2) \,|\, |(-v,v)|_G = \sqrt{2}|v|_g < \sqrt{2}\varepsilon\}$$

and $TM \cong N_M,\ (p,v) \mapsto \big((p,p),(-v,v)\big)$ the embedding (of pairs)

$$\begin{aligned} s'_M : (D_\varepsilon TM, D_\varepsilon TM - 0) &\to \big(M \times M, M \times M - \Delta(M)\big) \\ (p,v) &\mapsto \big(\exp(p,-v), \exp(p,v)\big) = \mathrm{Exp}\big((p,p,-v,v)\big) \end{aligned}$$

is then almost a tubular neighbourhood map (not yet defined on all of TM) and the image of s'_M is a tubular neighbourhood by the usual proof of the tubular neighbourhood theorem ([**Lee18**, Theorem 5.25] or [**Pet16**, Corollary 5.5.3] or [**BJ73**, Satz 12.11]). If we instead decompose (p,p,v,w) as

$$(p,p,v,w) = (p,p,0,w-v) + (p,p,v,v)$$

with $N' := \{(p,p,v,w) \in T(M \times M)_{|\Delta(M)} |\ v = 0\}$ then $TM \to N',\ (p,v) \mapsto (p,p,0,v)$ is a different embedding of N_M as a complement of $TM \cong T\Delta(M)$ in $(TM \times TM)_{|\Delta(M)}$. Exp restricted to $D_\varepsilon N'$ is also a diffeomorphism (for ε small enough) and the embedding (of pairs)

$$\begin{aligned} t'_M : (D_\varepsilon TM, D_\varepsilon TM - 0) &\to \big(M \times M, M \times M - \Delta(M)\big) \\ (p,v) &\mapsto \big(p, \exp(p,v)\big) = \mathrm{Exp}\big((p,p,0,v)\big) \end{aligned}$$

is also almost a tubular neighbourhood map ([**Pet16**, Proposition 5.5.1 (2)] or [**Lee18**, Proof of Lemma 6.14]). □

t'_M and s'_M are homotopic via the homotopy $H : TM \times [0,1] \to M^2,\ \big(\exp(p,-tv), \exp(p,v)\big)$. We have $H_1 = t'_M$ and $H_0 = s'_M$. We could have constructed a metric on $TM \times TM$ such that t'_M is the restriction of its exponential map to the orthogonal complement of TM with respect to that metric. However, the remark on page 118 of [**MS74**] shows that any two Riemannian metrics have homotopic exponential map. Note that we can use the same ε for both maps, as long as it is small enough such that both maps are embeddings. We can thus extend these two maps "simultaneously" using, for example, the fibre preserving diffeomorphism $\zeta_\varepsilon : TM \to D_\varepsilon TM,\ (p,v) \mapsto (p, \frac{\varepsilon v}{1+|v|_g})$. The compositions $s_M = s'_M \circ \zeta_\varepsilon$ and $t_M = t'_M \circ \zeta_\varepsilon$ are tubular neighbourhood maps.

Let $d_g(p,q)$ denote the distance in M between the points $p,q \in M$ defined by the metric g. We define the open neighbourhood $U_{M,\varepsilon}$ of $\Delta(M)$ inside M^2 by $U_{M,\varepsilon} := \{(p,q) \in M^2 |\ d_g(p,q) < \varepsilon\}$. In the following, we are going to use the above identifications $TM \cong N_M$ and view s_M, t_M as defined on elements $(p,v) \in N_M$. We have $U_{M,\varepsilon} = t_M(N_M) = t'_M(D_\varepsilon N_M)$ and $U_{M,2\varepsilon} = s_M(N_M) = s'_M(D_\varepsilon N_M)$ if ε is chosen small enough. In particular, s'_M, s_M, t'_M, t_M are open maps. (t'_M should be restricted to vectors v with $|v|_g$ smaller than the injectivity radius of (M,g) and for s'_M the norm $|v|_g$ should be smaller than half the injectivity radius of (M,g). The injectivity radius of a compact Riemannian manifold is positive ([**Lee18**, Lemma 6.16])).

Let ρ, with $0 < \rho < \infty$, denote the injectivity radius of (M,g). In [**HW18**, Lemma 2.1] the authors postulate the existence of a smooth map h that "pushes points" on M: The map

$$h : U_{M,\rho} \times M \subset M \times M \times M \to M \tag{2.2}$$

should satisfy that

(1) $h(p,q) := h(p,q,\cdot) : M \to M$ is a diffeomorphism if $d_g(p,q) < \frac{\rho}{14}$.
(2) $h(p,q)(p) = q$ if $d_g(p,q) < \frac{\rho}{14}$, i.e. it pushes the first point to the second if the two points a close enough.
(3) $h(p,p) = id_M$.

An explicit example of such a map is also given there, namely:

$$h(p,q)(x) := \begin{cases} \exp\Big(p, \exp_p^{-1}(x) + \mu\big(d_g(p,x)\big)\exp_p^{-1}(q)\Big) & \text{if } d_g(p,x) \le \rho/2 \\ w & \text{if } d_g(p,x) \ge \rho/2 \end{cases}$$

where $\mu : [0,\infty) \to \mathbb{R}$ is a smooth "cut-off" function that is constant 1 near 0, constant 0 on $[\frac{\rho}{3},\infty)$ and decays appropriately in between.
We are now going to use the map h to define explicit tubular neighbourhoods and tubular neighbourhood maps for $\mathcal{F}$ in Λ^2.

Tubular neighbourhood of Hingston and Wahl. In [**HW18**, Proposition 2.2] they use t'_M and construct $t_{\mathcal{F}}$ as follows:
We have $D_\varepsilon {ev_0}^*(N_M) = {ev_0}^*(D_\varepsilon N_M)$ if we equip ${ev_0}^*(N_M)$ with the pullback metric. We choose $\varepsilon < \frac{\rho}{14}$. The embedding $t'_{\mathcal{F}} : {ev_0}^*(D_\varepsilon N_M) \hookrightarrow \Lambda^2$ is defined by

$$t'_{\mathcal{F}}\big((\gamma,\delta),(x,v)\big) := \big(\gamma, \lambda(\delta,v)\big)$$

and uses the map h from above that "pushes points" on M: Here, the curve λ is defined to be

$$\lambda(\delta,v)(t) := h\big(\delta(0), \exp(\delta(0),v)\big)\big(\delta(t)\big) = h\big(t'_M(\delta(0),v)\big)\big(\delta(t)\big).$$

Since $|v| < \varepsilon < \frac{\rho}{14}$, $d_g\big(\delta(0), \exp(\delta(0),v)\big) = d_g\big(\delta(0),\lambda(0)\big) < \varepsilon < \frac{\rho}{14}$ and $h\big(\delta(0),\exp(\delta(0),v)\big) : M \to M$ is a diffeomorphism. Thus, instead of starting at $\delta(0)$, the curve λ starts at $\exp(\delta(0),v)$. "δ is diffeomorphically pushed away from γ".

Proposition 2.3. *(see* [**HW18**, Proposition 2.2]*) Let (M,g) be a compact Riemannian manifold with injectivity radius ρ and let $0 < \varepsilon < \frac{\rho}{14}$. Then the map $t'_{\mathcal{F}} : {ev_0}^*(D_\varepsilon N_M) \hookrightarrow \Lambda^2$ is an open embedding with image $U_{\mathcal{F},\varepsilon} := \{(\gamma,\delta) \in \Lambda^2 |\, d_g\big(\gamma(0),\delta(0)\big) < \varepsilon\}$ which is an open neighbourhood of $\mathcal{F}$. Moreover, the diagram*

$$\begin{tikzcd} {ev_0}^*(D_\varepsilon N_M) \arrow[r, hook, "t'_{\mathcal{F}}"] \arrow[d, "pr_{N_M}"'] & \Lambda \times \Lambda \arrow[d, "ev_0 \times ev_0"] \\ D_\varepsilon N_M \arrow[r, hook, "t'_M"] & M \times M \end{tikzcd} \tag{2.3}$$

commutes and $(ev_0 \times ev_0)(U_{\mathcal{F},\varepsilon}) = U_{M,\varepsilon}$.

Proof. From the definitions it follows that

- λ is continuous: $\lambda(\delta,v) = (\tilde{h} \circ t'_M \circ pr_{N_M})\big((\gamma,\delta),(x,v)\big) \circ \delta$, where $\tilde{h} : U_{M,\rho} \to C^0(M,M)$, $(p,q) \mapsto h(p,q,\cdot)$ is continuous since h is continuous and the composition is continuous since S^1 and M are (locally) compact ([**LS15**, Folgerung 4.23]).

$$\begin{tikzcd} {ev_0}^*(D_\varepsilon N_M) \arrow[r, "(\tilde{h}\circ t'_M\circ pr_{N_M}, pr_2\circ i_{\mathcal{F}}\circ pr_{\mathcal{F}})"] \arrow[ddr, "\lambda"'] & C^0(M,M)\times\Lambda M \arrow[d, hook] \\ & C^0(M,M)\times C^0(S^1,M) \arrow[d, "\circ"] \\ & \Lambda M \subset C^0(S^1,M). \end{tikzcd}$$

- $\lambda(\delta, v)$ is again an H^1-map, i.e. $\lambda(\delta, v) \in \Lambda M$.

It is clear that $t_{\mathcal{F}}$ maps into $U_{\mathcal{F},\varepsilon} = \{(\gamma,\delta) \in \Lambda^2 |\, d_g\big(\gamma(0), \delta(0)\big) < \varepsilon\}$. We also have $(ev_0 \times ev_0)(U_{\mathcal{F},\varepsilon}) = U_{M,\varepsilon}$ and the diagram

$$\begin{tikzcd} {ev_0}^*(D_\varepsilon N_M) \arrow[r, hook, "t'_{\mathcal{F}}"] \arrow[d, "pr_{N_M}"'] & \Lambda\times\Lambda \arrow[d, "ev_0\times ev_0"] \\ D_\varepsilon N_M \arrow[r, hook, "t'_M"] & M\times M \end{tikzcd}$$

commutes.
We still have to show that $t'_{\mathcal{F}}$ is a homeomorphism onto its image. To this end, in [**HW18**] they explicitly give the continuous inverse $\kappa : U_{\mathcal{F},\varepsilon} \to {ev_0}^*(D_\varepsilon N_M)$ defined by

$$\kappa(\gamma, \lambda) := (\gamma, \delta, v) = (\gamma, \delta, \gamma(0), v)$$

where

$$\delta(t) := h\big(\gamma(0), \lambda(0)\big)^{-1}\big(\lambda(t)\big),$$
$$v := \exp(\gamma(0), \cdot)^{-1}\big(\lambda(0)\big).$$

Since by the properties of h, $\delta(0) = h\big(\gamma(0), \lambda(0)\big)^{-1}\big(\lambda(0)\big) = \gamma(0)$, we then have that

$$\begin{aligned} t'_{\mathcal{F}}\big(\kappa(\gamma,\lambda)\big) &= t'_{\mathcal{F}}(\gamma,\delta,v) = \Big(\gamma, h\big(\delta(0), \exp(\delta(0), v)\big)\big(\delta(t)\big)\Big) \\ &= \Big(\gamma, h\big(\gamma(0), \lambda(0)\big)\big(\delta(t)\big)\Big) = \Big(\gamma, h\big(\gamma(0), \lambda(0)\big)\big(h\big(\gamma(0), \lambda(0)\big)^{-1}\big(\lambda(t)\big)\big)\Big) \\ &= (\gamma, \lambda). \end{aligned}$$

and similarly for $\kappa \circ t'_{\mathcal{F}}$.
The continuity of κ can been seen as follows:

- $\delta = \big(\tilde{h} \circ ev_0 \times ev_0(\gamma, \lambda)\big)^{-1} \circ \lambda$, where $\tilde{h} \circ (ev_0 \times ev_0)(\gamma, \lambda)$ is in fact an automorphism of M and the inversion of the group $Aut(M) \subset C^0(M, M)$ of automorphisms of M is continuous since M is compact ([**LS15**, Satz 5.1]).
- To show that $v = v\big(\gamma(0), \lambda(0)\big)$ is continuous we must show that $U_{M,\varepsilon} = (ev_0 \times ev_0)(U_{\mathcal{F},\varepsilon}) \to TM$, $(p, q) \mapsto (\exp_p)^{-1}(q)$ is continuous. Here $\exp_p = \exp(p, \cdot) : T_pM \to M$, which is defined on all of T_pM since M is compact. It suffices to note that $(p, v) = \big(p, (\exp_p)^{-1}(q)\big)$ in the preimage of (p, q) under the diffeomorphism

$$t'_M : D_\varepsilon N_M \to U_{M,\varepsilon}, \text{ so we already know that } (p,q) \mapsto (\exp_p)^{-1}(q) \text{ is continuous.}$$

We have thus shown: $t'_{\mathcal{F}}$ is an continuous map. Its image lies in $U_{\mathcal{F},\varepsilon}$. On the set $U_{\mathcal{F},\varepsilon}$ equipped with the subspace topology from Λ^2 there is a continuous map κ which maps into $ev_0{}^*(D_\varepsilon N_M)$ and is inverse to $t'_{\mathcal{F}}$. That is: $t'_{\mathcal{F}} : ev_0{}^*(D_\varepsilon N_M) \hookrightarrow U_{\mathcal{F},\varepsilon}$ is a continuous map with continuous inverse, hence a homeomorphism and so $t'_{\mathcal{F}} : ev_0{}^*(D_\varepsilon N_M) \hookrightarrow \Lambda^2$ is an embedding. Since, by definition, $U_{\mathcal{F},\varepsilon} = (ev_0 \times ev_0)^{-1}(U_{M,\varepsilon})$, $t'_{\mathcal{F}}$ is an embedding with open image $U_{\mathcal{F},\varepsilon}$ and thus is an open map. ☐

The fibre preserving diffeomorphism $\zeta_\varepsilon : ev_0{}^*(N_M) \to ev_0{}^*(D_\varepsilon N_M),\ (\gamma,\delta,v) \mapsto (\gamma,\delta,\frac{\varepsilon v}{1+|v|_g})$, can be used to extend $t'_{\mathcal{F}}$: We define

$$t_{\mathcal{F}} := t'_{\mathcal{F}} \circ \zeta_\varepsilon : ev_0{}^*(N_M) \to \Lambda^2.$$

$t_{\mathcal{F}}$ restricts to the inclusion of $\mathcal{F}$ on the zero section $z_{\mathcal{F}}$, as

$$\begin{tikzcd} ev_0{}^*(N_M) \arrow[r, hook, "t_{\mathcal{F}}"] & \Lambda \times \Lambda \\ \mathcal{F} \arrow[u, hook, "z_{\mathcal{F}}"] \arrow[ur, hook, "i_{\mathcal{F}}"'] & \end{tikzcd}$$

commutes, since $h(p,p) = id_M$. It hence is a tubular neighbourhood map whose image is $U_{\mathcal{F},\varepsilon}$.

Symmetrized tubular neighbourhood. For our purposes we will also need a different tubular neighbourhood: we will have to "push points symmetrically". By that we mean that we will have to push both curves γ and δ in opposite directions away from the common starting point $\gamma(0) = \delta(0)$.
We therefore define the continuous map $s'_{\mathcal{F}} : ev_0{}^*(D_\varepsilon N_M) \hookrightarrow \Lambda^2$ by

$$s'_{\mathcal{F}}\big((\gamma,\delta),(x,v)\big) = s'_{\mathcal{F}}(\gamma,\delta,v) := \big(\lambda(\gamma,-v),\lambda(\delta,v)\big) =: (\alpha,\beta).$$

We choose $\varepsilon < \frac{\rho}{14}$. Now $\alpha = \lambda(\gamma,-v)$ starts at $\exp(\gamma(0),-v)$ and $\beta = \lambda(\delta,v)$ at $\exp(\delta(0),v)$. The distance between these new starting points is now

$$\begin{aligned} d_g\big(\alpha(0),\beta(0)\big) &= d_g\big(\exp(\gamma(0),-v),\exp(\delta(0),v)\big) \\ &\le d_g\big(\exp(\gamma(0),-v),\gamma(0)\big) + d_g\big(\exp(\delta(0),v),\delta(0)\big) < 2\varepsilon < 2\frac{\rho}{14}. \end{aligned}$$

PROPOSITION 2.4. *Let (M,g) be a compact Riemannian manifold with injectivity radius ρ and let $0 < \varepsilon < \frac{\rho}{14}$. Then the map $s'_{\mathcal{F}} : ev_0{}^*(D_\varepsilon N_M) \hookrightarrow \Lambda^2$ is an open embedding with image $U_{\mathcal{F},2\varepsilon} := \{(\gamma,\delta) \in \Lambda^2 |\ d_g\big(\gamma(0),\delta(0)\big) < 2\varepsilon\}$ which is an open neighbourhood of $\mathcal{F}$. Moreover, the diagram*

$$\begin{tikzcd} ev_0{}^*(D_\varepsilon N_M) \arrow[r, hook, "s'_{\mathcal{F}}"] \arrow[d, "pr_{N_M}"'] & \Lambda \times \Lambda \arrow[d, "ev_0 \times ev_0"] \\ D_\varepsilon N_M \arrow[r, hook, "s'_M"] & M \times M \end{tikzcd}$$

commutes and $(ev_0 \times ev_0)(U_{\mathcal{F},2\varepsilon}) = U_{M,2\varepsilon}$.

PROOF. $s'_{\mathcal{F}}$ clearly maps into $U_{\mathcal{F},2\varepsilon} = \{(\gamma,\delta) \in \Lambda^2 |\ d_g\big(\gamma(0) - \delta(0)\big) < 2\varepsilon\}$ and $(ev_0 \times ev_0)(U_{\mathcal{F},2\varepsilon}) = U_{M,2\varepsilon}$. The diagram

$$\begin{tikzcd} {ev_0}^*(D_\varepsilon N_M) \arrow[r, hook, "s'_{\mathcal{F}}"] \arrow[d, "pr_{N_M}"'] & \Lambda \times \Lambda \arrow[d, "ev_0 \times ev_0"] \\ D_\varepsilon N_M \arrow[r, hook, "s'_M"] & M \times M \end{tikzcd}$$

is easily seen to commute. Using the unique minimal geodesic $c_{\alpha(0),\beta(0)} : [0,1] \to M$ from $\alpha(0)$ to $\beta(0)$ (these points are less than the injectivity radius apart), we define an inverse $\mu : U_{\mathcal{F},2\varepsilon} \to {ev_0}^*(D_\varepsilon N_M)$ by

$$\mu(\alpha,\beta) := (\gamma,\delta,v) = \big(\gamma,\delta,c_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big),v\big)$$

where

$$\begin{aligned} \gamma(t) &:= h\big(c_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big),\alpha(0)\big)^{-1}\big(\alpha(t)\big), \\ \delta(t) &:= h\big(c_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big),\beta(0)\big)^{-1}\big(\beta(t)\big), \\ v &:= \tfrac{1}{2}\dot{c}_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big). \end{aligned}$$

As above, we have $\gamma(0) = c_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big) = \delta(0)$ as required.
Let us check that μ is indeed an inverse for $s'_{\mathcal{F}}$: For points $a, b \in M$ with $d_g(a,b) < 2\varepsilon < \rho = \rho_g$, there is a unique vector $v_b \in T_aM$ with $|v_b|_g < 2\varepsilon$ and $c_{a,b} := \big([0,1] \ni t \mapsto \exp(a, tv_b)\big)$ is the unique minimizing geodesic between a and b ([**Pet16**, Theorem 5.5.4]). We define $p := c_{a,b}(\frac{1}{2})$ and $v := \frac{1}{2}\dot{c}_{a,b}(\frac{1}{2})$, then $a := \exp(p,-v)$ and $b := \exp(p,v)$ with $|v|_g = \frac{1}{2}|v_b|_g < \varepsilon$. This means $s'_M(p,v) = (a,b)$ and it is the only point with this property in $D_\varepsilon N_M$ since $\varepsilon < \frac{\rho}{2}$.
We first show that

$$\mu \circ s'_{\mathcal{F}}(\gamma,\delta,v) = (\gamma,\delta,v).$$

Let $s'_{\mathcal{F}}(\gamma,\delta,v)$ be (α,β). Then, applying μ yields for the first coordinate

$$\begin{aligned} h\big(c_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big),\alpha(0)\big)^{-1}\big(\alpha(t)\big) &= h\big(\gamma(0),\exp(\gamma(0),-v)\big)^{-1}\big(\lambda(\gamma,-v)(t)\big) \\ &= h\big(\gamma(0),\exp(\gamma(0),-v)\big)^{-1}\Big(h\big(\gamma(0),\exp(\gamma(0),-v)\big)\big(\gamma(t)\big)\Big) = \gamma(t) \end{aligned}$$

since $d_g\big(\gamma(0),\exp(\gamma(0),-v)\big) < \varepsilon < \frac{\rho}{14}$. Similarly for the second coordinate. That $\frac{1}{2}\dot{c}_{\alpha,\beta}(\frac{1}{2})$ equals v follows from the preceding discussion.
Now we have to show that

$$s'_{\mathcal{F}} \circ \mu(\alpha,\beta) = (\alpha,\beta).$$

Let $\mu(\alpha,\beta)$ be (γ,δ,v). Then

$$\begin{aligned} s'_{\mathcal{F}}(\gamma,\delta,v) &= \big(\lambda(\gamma,-v),\lambda(\delta,v)\big) = \Big(h\big(\gamma(0),\exp(\gamma(0),-v)\big)\circ\gamma, h\big(\delta(0),\exp(\delta(0),v)\big)\circ\delta\Big) \\ &= \Big(h\big(\gamma(0),\alpha(0)\big)\circ h\big(\gamma(0),\alpha(0)\big)^{-1}\circ\alpha, h\big(\delta(0),\beta(0)\big)\circ h\big(\delta(0),\beta(0)\big)^{-1}\circ\beta\big)\Big) \\ &= (\alpha,\beta), \end{aligned}$$

where the second equality holds since $\exp(\gamma(0),-v) = \exp\Big(c_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big), -\tfrac{1}{2}\dot{c}_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big)\Big) = \alpha(0)$ and similarly $\exp(\delta(0),v) = \beta(0)$.
Lastly, we are left with showing that μ is continuous: That γ and δ are continuous follows as in the case of κ. Smoothness of $\big(c_{\alpha(0),\beta(0)}\big(\tfrac{1}{2}\big),v\big)$ is similarly proved as in the case of κ as

well. Here we first project (α, β) to $\big(\alpha(0), \beta(0)\big) \in U_{M,2\varepsilon}$ via $ev_0 \times ev_0$ and then apply the inverse of s'_M which maps the pair of points (a, b) to $\big(p, \exp_p^{-1}(b)\big) = \big(p, -\exp_p^{-1}(a)\big)$ where p is the midpoint of the geodesic $c_{a,b}$. □

Again, we have shown that $s'_{\mathcal{F}}$ is an open embedding with (open) image $U_{\mathcal{F},2\varepsilon}$. As above we can extend it to all of ${ev_0}^*(N_M)$ and as above we have that

$$\begin{tikzcd} {ev_0}^*(N_M) \arrow[r, hook, "s_{\mathcal{F}}"] & \Lambda \times \Lambda \\ \mathcal{F} \arrow[u, hook, "z_{\mathcal{F}}"] \arrow[ur, hook, "i_{\mathcal{F}}"'] & \end{tikzcd}$$

commutes. Hence $s_{\mathcal{F}} : {ev_0}^*(N_M) \hookrightarrow \Lambda^2$ is a tubular neighbourhood map whose image $U_{\mathcal{F},2\varepsilon}$ is an open tube around $\mathcal{F}$.

3. Relation to the Pontrjagin product

Let us look again at the loop fibration $\Omega \overset{j}{\hookrightarrow} \Lambda \xrightarrow{ev_0} M$. Later we are going to make use of

Proposition 3.1. [**CS99**, Proposition 3.4] *Let M be a connected, compact, oriented n-dimensional manifold.*

(1) The homomorphism $ev_{0*} : H_*(\Lambda; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ of graded modules is an algebra homomorphism from the Chas-Sullivan algebra to the intersection algebra.
(2) The homomorphism $j_! : H_*(\Lambda; \mathbb{Z}) \to H_{*-n}(\Omega; \mathbb{Z})$ of graded modules is an algebra homomorphism from the Chas-Sullivan algebra to the Pontrjagin algebra.

(Compare [**AS06**, Proposition 2.4])

More precisely, we are going to use the following three relations given in [**GH09**, Section 9.3] or in [**HR13**, Section 6, Equations 27-29] :

$$j_!(a * b) = j_!(a) \star j_!(b) \tag{3.1}$$

$$j_*(y) * a = j_*\big(y \star j_!(a)\big) \tag{3.2}$$

$$j_*\big(j_!(a)\big) = A * a \tag{3.3}$$

where A is a generator of $H_0(\Lambda)$ and $\star$ denotes the Pontrjagin product. The third equation follows from the second noting that in the Pontrjagin algebra the class of a constant loop is a unit.

The rest of this section is devoted to partial proofs of the above. These proofs can be skipped since we are only going to use the three equations above.

Partial proof. We assume M to be oriented. Throughout the proof $H_*(\cdot)$ will denote singular homology with integer coefficients.

(1) Since $\Delta : M \to M^2$ embeds M smoothly into $M^2 := M \times M$ with codimension n, there is, as above, a Gysin map

$$\Delta_! : H_i(M^2) \to H_{i-n}(M).$$

We now show that the pullback square

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook, "i_{\mathcal{F}}"] \arrow[d, "ev_0"'] & \Lambda^2 \arrow[d, "ev_0\times ev_0"] \\
M \arrow[r, hook, "\Delta"'] & M^2
\end{tikzcd}$$

implies that the diagram

$$\begin{tikzcd}
H_{i-n}(\mathcal{F}) \arrow[d, "(ev_0)_*"'] & H_i(\Lambda^2) \arrow[l, "i_{\mathcal{F}!}"'] \arrow[d, "(ev_0\times ev_0)_*"] \\
H_{i-n}(M) & H_i(M^2) \arrow[l, "\Delta_!"']
\end{tikzcd} \tag{3.4}$$

commutes: Of course this holds due to what we have previously shown, namely that $N_{\mathcal{F}} \cong {ev_0}^*(N_M)$. Recall the commutative diagram

$$\begin{tikzcd}
N_{\mathcal{F}} \arrow[dr, "g"', "\cong"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& {ev_0}^*(N_M) \arrow[r, "pr_{N_M}"] \arrow[d] & N_M \arrow[d] \\
& \mathcal{F} \arrow[r, "ev_0"] & M.
\end{tikzcd}$$

Let τ_M be the Thom class of the embedding $M \hookrightarrow M^2$. Naturality of the cap product and naturality of the maps in the exact homology sequence of pairs then give the above homology diagram: Naturality of the cap product is the commutativity of the diagram

$$\begin{tikzcd}
H_i(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}) \arrow[r, "g_*"] \arrow[d, "\cap g^*\left({pr_{N_M}}^*(\tau_M)\right)"] & H_i\left({ev_0}^*(N_M), {ev_0}^*(N_M) - \mathcal{F}\right) \arrow[r, "{pr_{N_M}}_*"] \arrow[d, "\cap {pr_{N_M}}^*(\tau_M)"] & H_i(N_M, N_M - M) \arrow[d, "\cap\tau_M"] \\
H_{i-n}(N_{\mathcal{F}}) \arrow[r, "g_*"'] & H_{i-n}\left({ev_0}^*(N_M)\right) \arrow[r, "{pr_{N_M}}_*"'] & H_{i-n}(N_M).
\end{tikzcd}$$

Let us define $\tau_{\mathcal{F}} := (pr_{N_M} \circ g)^*(\tau_M) = f^*(\tau_M)$.
Let $t_M : N_M \to M^2$ be a tubular neighbourhood map of $\Delta(M)$ in M^2 with image $U_M := t_M(N_M)$. By [**Lan99**, Chapter IV, Sections 5 and 6 and Chapter VII, Section 4] the existence of tubular neighbourhoods is also guaranteed for Hilbert manifolds. Thus we similarly have a tubular neighbourhood map $t_{\mathcal{F}} : N_{\mathcal{F}} \to \Lambda^2$ with image $U_{\mathcal{F}} := t_{\mathcal{F}}(N_{\mathcal{F}})$. The following two vertical compositions are then Gysin maps (they

do not depend on the particular choice of tubular neighbuorhood maps):

$$
\begin{tikzcd}
H_i(\Lambda^2) \arrow[r, "(ev_0\times ev_0)_*"] \arrow[d] \arrow[ddddd, bend right, "i_{\mathcal{F}!}"'] & H_i(M^2) \arrow[d] \arrow[ddddd, bend left, "\Delta_!"] \\
H_i(\Lambda^2, \Lambda^2 - \mathcal{F}) \arrow[r, "(ev_0\times ev_0)_*"] \arrow[d, "\cong"'] & H_i\big(M^2, M^2 - \Delta(M)\big) \arrow[d, "\cong"'] \\
H_i(U_{\mathcal{F}}, U_{\mathcal{F}} - \mathcal{F}) \arrow[r, dashed] \arrow[d, "\cong"', "(t_{\mathcal{F}*})^{-1}"] & H_i(U_M, U_M - \Delta(M)) \arrow[d, "\cong"', "(t_{M*})^{-1}"] \\
H_i(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}) \arrow[r, "f_*"'] \arrow[d, "\cong"', "\cap\tau_{\mathcal{F}}"] & H_i(N_M, N_M - M) \arrow[d, "\cong"', "\cap\tau_M"] \\
H_{i-n}(N_{\mathcal{F}}) \arrow[r, "f_*=(pr_{N_M}\circ g)_*"'] \arrow[d, "\cong"'] & H_{i-n}(N_M) \arrow[d, "\cong"'] \\
H_{i-n}(\mathcal{F}) \arrow[r, "ev_{0*}"'] & H_{i-n}(M).
\end{tikzcd}
$$

In the above diagram commutativity is clear except at the dashed arrow. Thus we are left to show that there are tubular neighbourhoods which are compatible, i.e. we have to show that the diagram

$$
\begin{tikzcd}
N_{\mathcal{F}} \arrow[r, hook, "t_{\mathcal{F}}"] \arrow[d, "f"'] & \Lambda^2 \arrow[d, "ev_0\times ev_0"] \\
N_M \arrow[r, hook, "t_M"] & M^2
\end{tikzcd}
$$

commutes, at least up to homotopy. We have already constructed explicit maps $t_{\mathcal{F}}$ and t_M for which the diagram commutes in the last section and so we have already proved the commutativity of the diagram above which is diagram 3.4.

Concatenation $\phi = \phi_{\frac{1}{2}} : \mathcal{F} \to \Lambda$ satisfies

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[d, "ev_0"'] & \Lambda \arrow[dl, "ev_0"] \\
M &
\end{tikzcd}
$$

and thus, together with diagram 3.4, we get the commutative diagram

$$\begin{tikzcd}
H_i(\Lambda) \times H_j(\Lambda) \arrow[r, "\times"] \arrow[d, "ev_{0*} \times ev_{0*}"] \arrow[rrr, bend left, "*'"] & H_{i+j}(\Lambda \times \Lambda) \arrow[r, "i_{\mathcal{F}!}"] \arrow[d, "(ev_0 \times ev_0)_*"] & H_{i+j-n}(\mathcal{F}) \arrow[r, "\phi_*"] \arrow[d, "ev_{0*}"'] & H_{i+j-n}(\Lambda) \arrow[dl, "ev_{0*}"] \\
H_i(M) \times H_j(M) \arrow[r, "\times"] \arrow[rr, bend right, "\bullet'"'] & H_{i+j}(M \times M) \arrow[r, "\Delta_!"] & H_{i+j-n}(M) &
\end{tikzcd}$$

where $*'$ and $\bullet'$ denote the not-sign-corrected Chas-Sullivan product and intersection product respectively. This proves that $ev_{0*} : \big(H_*(\Lambda), *\big) \to \big(H_*(M), \bullet\big)$, where $\bullet$ denotes the intersection product, is an algebra homomorphism up to sign. We leave it to the reader to check that signs match, since we are not going to need the correct signs here. References here are [**Bre93**, Chaper VI, Section 11], [**MS74**, Chapter 11], [**HW18**, Appendix B], [**Dol95**, Chapter VIII, Section 13].

(2) We now consider the inclusion $j_p : \Omega_p \hookrightarrow \Lambda$ of the loops based at a given point $p \in M$. Here $\Omega_p := \{\gamma \in \Lambda | \, \gamma(0) = p\} = ev_0^{-1}(p)$. We again use that $ev_0 : \Lambda \to M$ and $ev_0 \times ev_0 : \Lambda^2 \to M^2$ are smooth submersions. Then, if we consider the inclusions of points $\{p\} \hookrightarrow M$, $\{(p,p)\} \hookrightarrow M^2$, it follows that $ev_0^{-1}(\{p\}) = \Omega_p \subset \Lambda$ and $(ev_0 \times ev_0)^{-1}(\{(p,p)\}) = \Omega_p^2 \subset \Lambda^2$ are submanifolds of codimension n and $2n$ respectively. We fix some base point $p \in M$ and just write j and Ω for j_p, Ω_p from now on. The commutative diagrams

$$\begin{tikzcd}
\Omega \arrow[r, hook, "j"] \arrow[d, "ev_0"'] & \Lambda \arrow[d, "ev_0"] \\
\{p\} \arrow[r, hook] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^2 \arrow[r, hook, "j \times j"] \arrow[d, "ev_0 \times ev_0"'] & \Lambda^2 \arrow[d, "ev_0 \times ev_0"] \\
\{(p,p)\} \arrow[r, hook] & M^2
\end{tikzcd}$$

are pullback squares and show that $\Omega \subset \Lambda$ and $\Omega^2 \subset \Lambda^2$ have trivial normal bundles since they are isomorphic to pullbacks of (normal) bundles over points. This assures the existence of Thom classes for any coefficients. If τ_Ω denotes a Thom class of the normal bundle of $\Omega \subset \Lambda$, then, obviously $\tau_\Omega \times \tau_\Omega$ is a Thom class for $\Omega^2 \subset \Lambda^2$. Let us denote the normal bundle of Ω^2 inside Λ^2 by E. We have

$$E \cong (ev_0)^*(T_pM) \times (ev_0)^*(T_pM)$$

and can take this as a definition of that bundle. We set

$$\tau_E := \tau_\Omega \times \tau_\Omega.$$

So E is a vector bundle $E \to \Omega^2$ of rank $2n$ that embeds into Λ^2 as an open neighboorhood of Ω^2.

Since $j \times j$ maps into $\mathcal{F}$, the diagram

$$\begin{tikzcd} & \mathcal{F} \arrow[dr, hook, "i_{\mathcal{F}}"] & \\ \Omega \times \Omega \arrow[ur, hook, "j_0"] \arrow[rr, hook, "j\times j"] & & \Lambda \times \Lambda \end{tikzcd}$$

commutes. Here j_0 denotes the restriction of $j \times j$. This shows that $\Omega \times \Omega$ is a submanifold of $\mathcal{F}$. It has codimension n and trivial normal bundle. We now show that this implies that

$$\begin{tikzcd} & H_{i-n}(\mathcal{F}) \arrow[dl, "j_{0!}"'] & \\ H_{i-2n}(\Omega \times \Omega) & & H_i(\Lambda \times \Lambda) \arrow[ul, "i_{\mathcal{F}!}"'] \arrow[ll, "(j\times j)_!"'] \end{tikzcd}$$

commutes up to sign: For clarity, we denoted the different evaluation maps by

$$\begin{aligned} e_{\Omega^2} &:= ev_0 \times ev_0 : \Omega^2 \to \{(p,p)\}, \\ e_{\mathcal{F}} &:= ev_0 \times ev_0 : \mathcal{F} \to \Delta(M), \\ e_{\Lambda^2} &:= ev_0 \times ev_0 : \Lambda^2 \to M^2. \end{aligned}$$

Putting the above information together we get the following commutative diagram

$$\begin{tikzcd} \Omega^2 \arrow[r, hook, "j_0"'] \arrow[d, "e_{\Omega^2}"'] \arrow[rr, hook, bend left, "j\times j"] & \mathcal{F} \arrow[r, hook, "i_{\mathcal{F}}"'] \arrow[d, "e_{\mathcal{F}}"] & \Lambda^2 \arrow[d, "e_{\Lambda^2}"] \\ \{(p,p)\} \arrow[r, hook, "i^{\Delta(M)}_{\{(p,p)\}}"] \arrow[rr, hook, bend right, "i_{\{(p,p)\}}"'] & \Delta(M) \arrow[r, hook, "i_{\Delta(M)}"] & M^2 \end{tikzcd}$$

and the transversality of $ev_0 \times ev_0$ (w.r.t. any submanifold of Λ^2) implies

$$N_{\mathcal{F}} \cong (e_{\mathcal{F}})^* \big(N_{\Delta(M)}\big) = (e_{\mathcal{F}})^* \big(T(M^2)|_{\Delta(M)} / T\Delta(M)\big)$$

and

$$
\begin{aligned}
E &\cong (ev_0)^*(T_pM) \times (ev_0)^*(T_pM) \\
&\cong j^*\big((ev_0)^*(TM)\big) \times j^*\big((ev_0)^*(TM)\big) \\
&\cong (j \times j)^*\Big((e_{\Lambda^2})^*\big(T(M^2)\big)\Big) \\
&\cong (e_{\Omega^2})^*\Big((i_{\{(p,p)\}})^*\big(T(M^2)\big)\Big) \\
&\cong (e_{\Omega^2})^*\big(T(M^2)|_{\{(p,p)\}}\big) = (e_{\Omega^2})^*\big(N_{\{(p,p)\}}\big) \\
&\cong (e_{\Omega^2})^*\Big((i^{\Delta(M)}_{\{(p,p)\}})^*\big(T(M^2)|_{\Delta(M)}\big)\Big) \\
&\cong j_0^*\Big((e_{\mathcal{F}})^*\big(T(M^2)|_{\Delta(M)}\big)\Big) \\
&\cong j_0^*\Big((e_{\mathcal{F}})^*\big(N_{\Delta(M)} \oplus T\Delta(M)\big)\Big) \\
&\cong j_0^*\Big(N_{\mathcal{F}} \oplus (e_{\mathcal{F}})^*\big(T\Delta(M)\big)\Big) \\
&\cong j_0^*\Big(N_{\mathcal{F}}) \oplus j_0^*\big((e_{\mathcal{F}})^*\big(T\Delta(M)\big)\big)\Big) \\
&\cong j_0^*\big(N_{\mathcal{F}}\big) \oplus (e_{\Omega^2})^*\Big((i^{\Delta(M)}_{\{(p,p)\}})^*\big(T\Delta(M)\big)\Big) \\
&\cong j_0^*\big(N_{\mathcal{F}}\big) \oplus (e_{\Omega^2})^*\big(T_{\{p,p\}}\Delta(M)\big) \\
&\cong j_0^*\big(N_{\mathcal{F}}\big) \oplus (e_{\Omega^2})^*(T_pM) \\
&\cong j_0^*\big(N_{\mathcal{F}}\big) \oplus (\Omega^2 \times \mathbb{R}^n).
\end{aligned} \tag{3.5}
$$

Let $p_{\Omega^2} : N_{\Omega^2} \to \Omega^2$ be the normal bundle of $\Omega^2 \subset \mathcal{F}$. We have that N_{Ω^2} is bundle isomorphic to $(e_{\Omega^2})^*\big(T_{\{p,p\}}\Delta(M)\big)$, so the above says

$$E \cong W := j_0^*\big(N_{\mathcal{F}}\big) \oplus N_{\Omega^2}.$$

That is, the normal bundle E of $\Omega^2 \subset \Lambda^2$ is the sum of the normal bundles $p_{\mathcal{F}} : N_{\mathcal{F}} \to \mathcal{F}$ of $\mathcal{F} \subset \Lambda^2$ and $p_{\Omega^2} : N_{\Omega^2} \to \Omega^2$ of $\Omega^2 \subset \mathcal{F}$, or, more precisely, isomorphic to the Whitney sum of the pullback bundle $p : {j_0}^*(N_{\mathcal{F}}) \to \Omega^2$ with the bundle N_{Ω^2}. The Whitney sum W is a pullback:

$$
\begin{array}{ccc}
W & \xrightarrow{pr_{{j_0}^*(N_{\mathcal{F}})}} & {j_0}^*(N_{\mathcal{F}}) \\
{\scriptstyle pr_{N_{\Omega^2}}}\downarrow & & \downarrow{\scriptstyle p} \\
N_{\Omega^2} & \xrightarrow[p_{\Omega^2}]{} & \Omega^2
\end{array}
$$

where the bundle projection is either of the two compositions along the edges.

Let now τ_0 be a Thom class of the bundle $p_{\Omega^2} : N_{\Omega^2} \to \Omega^2$. Let ${j_0}^*(\tau_{\mathcal{F}})$ denote the Thom class of the bundle ${j_0}^*(N_{\mathcal{F}}) \to \Omega^2$ induced by the class $\tau_{\mathcal{F}}$. It is not difficult to show that

$$\tau_W := (pr_{N_{\Omega^2}})^*(\tau_0) \cup (pr_{{j_0}^*(N_{\mathcal{F}})})^*\big({j_0}^*(\tau_{\mathcal{F}})\big)$$

is a Thom class for E ([**Hus94**, Chapter 17, Proposition 8.1] or [**GH09**, Appendix B.3]).

We also give a proof: Let $f : A \to X$ and $g : B \to X$ be vector bundles with Thom classes τ_A of A and τ_B of B. Then the Whitney sum $A \oplus B \xrightarrow{p_X} X$ is usually defined as the pullback

$$\begin{tikzcd} A \oplus B = X \times_{X \times X} A \times B \arrow[r, "\tilde{\Delta}"] \arrow[d, "p_X"'] & A \times B \arrow[d, "f \times g"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

and $\tilde{\Delta}^*(\tau_A \times \tau_B)$ is Thom class for it ([**AGP02**, Proposition 11.7.12]). It is readily seen that $h : A \times_X B \to X \times_{X \times X} (A \times B),\ (a, b) \mapsto (f(a) = g(b), a, b)$, is a vector bundle isomorphism over X so that, as stated above, the pullback

$$\begin{tikzcd} A \oplus B = A \times_X B \arrow[r, "pr_B \circ i"] \arrow[d, "pr_A \circ i"'] \arrow[dr] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

is also a valid definition of the Whitney sum. Here $i : A \times_X B \to A \times B$ is the inclusion and we see that

$$\begin{tikzcd} A \times_X B \arrow[rr, hook, "i"] \arrow[dr, "h"'] & & A \times B \arrow[r, "pr_{A,B}"] & A, B \\ & X \times_{X \times X} A \times B \arrow[ur, "\tilde{\Delta}"'] & & \end{tikzcd}$$

commutes. It follows that

$$\begin{aligned} h^*\big(\tilde{\Delta}^*(\tau_A \times \tau_B)\big) &= h^*\big(\tilde{\Delta}^*\big((pr_A)^*(\tau_A) \cup (pr_B)^*(\tau_B)\big)\big) \\ &= h^*\Big(\tilde{\Delta}^*\big((pr_A)^*(\tau_A)\big)\Big) \cup h^*\Big(\tilde{\Delta}^*\big((pr_B)^*(\tau_B)\big)\Big) \\ &= (pr_A \circ i)^*(\tau_A) \cup (pr_B \circ i)^*(\tau_B) \end{aligned}$$

by the definition of the cross product in cohomology and the properties of the cup product.

This proves that $\tau_W := (pr_{N_{\Omega^2}})^*(\tau_0) \cup (pr_{j_0{}^*(N_{\mathcal{F}})})^*\big(j_0{}^*(\tau_{\mathcal{F}})\big)$ is a Thom class for $W \cong E$.

Furthermore, let d_A, d_B be the ranks of these bundles. Let us use the notation $p_A := pr_A \circ i$ and $p_B := pr_B \circ i$. We then have

$$\begin{aligned} \mu_A &:= p_A^*(\tau_A) \in H^{d_A}(A \oplus B, A \oplus B - B), \\ \mu_B &:= p_B^*(\tau_B) \in H^{d_B}(A \oplus B, A \oplus B - A), \end{aligned}$$

and, as shown above, $\mu_A \cup \mu_B$ is a Thom class for the bundle $A \oplus B \to X$. It then follows that ([**Spa95**, page 254])

$$\begin{tikzcd}
H_k(A \oplus B, A \oplus B - X) \arrow[r, "\cap \mu_A"] \arrow[rr, bend left, "\cap(\mu_B \cup \mu_A)"] & H_{k-d_A}(A \oplus B, A \oplus B - A) \arrow[r, "\cap \mu_B"] & H_{k-d_A-d_B}(A \oplus B) \arrow[d, "\cong"] \\
& & H_{k-d_A-d_B}(X)
\end{tikzcd}$$

commutes. Note the reversal in order: $\mu_B \cup \mu_A = (-1)^{d_A d_B} \mu_A \cup \mu_B$.

Up to now, we have $W \cong E \cong (ev_0)^*(T_pM) \times (ev_0)^*(T_pM)$ (see (3.5)), so that

$$(pr_{N_{\Omega^2}})^*(\tau_0) \cup (pr_{{j_0}^*(N_{\mathcal{F}})})^* \big({j_0}^*(\tau_{\mathcal{F}})\big) = \tau_W = \pm \tau_\Omega \times \tau_\Omega.$$

where here "=" stand for "corresponding under the isomorphism".

We now construct a tubular neighbourhood map $E \hookrightarrow \Lambda^2$ for Ω^2: Let $t_{\mathcal{F}} : N_{\mathcal{F}} \hookrightarrow \Lambda^2$ be the tubular neighbourhood embedding of $\mathcal{F}$ in Λ^2 and let $t_{\Omega^2} : N_{\Omega^2} \hookrightarrow \mathcal{F}$ be the tubular embedding of Ω^2 in $\mathcal{F}$. We pull $N_{\mathcal{F}}$ back along t_{Ω^2}. The total space of the bundle ${t_{\Omega^2}}^*(N_{\mathcal{F}})$ then embeds via $t_{\mathcal{F}} \circ pr_{N_{\mathcal{F}}}$ into Λ^2 as an open neighbourhood of Ω^2 as the diagram

$$\begin{tikzcd}
{t_{\Omega^2}}^*(N_{\mathcal{F}}) \arrow[r, hook, "pr_{N_{\mathcal{F}}}"] \arrow[d] & N_{\mathcal{F}} \arrow[r, hook, "t_{\mathcal{F}}"] \arrow[d] & \Lambda^2 \\
N_{\Omega^2} \arrow[r, hook, "t_{\Omega^2}"] \arrow[d] & \mathcal{F} & \\
\Omega^2 & &
\end{tikzcd}$$

shows. Here we make use of the fact that as t_{Ω^2} is an open embedding, so is $pr_{N_{\mathcal{F}}}$: That $pr_{N_{\mathcal{F}}}$ is continuous and injective follows immediately, that is is open is assured e.g. by [**Hus94**, Chapter 2 Proposition 5.9]. (An open or closed injective continuous map is an embedding.)

Let $z_{\mathcal{F}} : \mathcal{F} \to N_{\mathcal{F}}$ and $z_{\Omega^2} : \Omega^2 \to N_{\Omega^2}$ be the zero sections of the two normal bundles. The commutative diagram

$$\begin{tikzcd}
{t_{\Omega^2}}^*(N_{\mathcal{F}}) \arrow[r, hook, "pr_{N_{\mathcal{F}}}"] & N_{\mathcal{F}} \arrow[r, hook, "t_{\mathcal{F}}"] & \Lambda^2 \\
N_{\Omega^2} \arrow[u, hook, "\widetilde{z_{\mathcal{F}}}"'] \arrow[r, hook, "t_{\Omega^2}"] & \mathcal{F} \arrow[u, hook, "z_{\mathcal{F}}"'] \arrow[ur, hook, "i_{\mathcal{F}}"'] & \\
\Omega^2 \arrow[u, hook, "z_{\Omega^2}"'] \arrow[ur, hook, "j_0"'] & &
\end{tikzcd}$$

then proves that $t_{\mathcal{F}} \circ pr_{N_{\mathcal{F}}} \circ \widetilde{z_{\mathcal{F}}} \circ z_{\Omega^2} = t_{\mathcal{F}} \circ z_{\mathcal{F}} \circ t_{\Omega^2} \circ z_{\Omega^2} = i_{\mathcal{F}} \circ j_0 = j \times j$. Here $\widetilde{z_{\mathcal{F}}}$ denotes the pullback of the section $z_{\mathcal{F}}$. Hence

$$\begin{tikzcd} {t_{\Omega^2}}^*(N_{\mathcal{F}}) \arrow[r, hook, "t_{\mathcal{F}} \circ pr_{N_{\mathcal{F}}}"] & \Lambda^2 \\ \Omega^2 \arrow[u, hook, "\widetilde{z_{\mathcal{F}}} \circ z_{\Omega^2}"] \arrow[ur, hook, "j \times j"'] & \end{tikzcd}$$

commutes and we only need to show that ${t_{\Omega^2}}^*(N_{\mathcal{F}}) \to N_{\Omega^2} \to \Omega^2$ is a vector bundle isomorphic to the Whitney sum $W = {j_0}^*(N_{\mathcal{F}}) \oplus N_{\Omega^2} \to \Omega^2 \cong E$.
By definition of t_{Ω^2}, we have $t_{\Omega^2} \circ z_{\Omega^2} = j_0$. This implies that ${j_0}^*(N_{\mathcal{F}}) = (t_{\Omega^2} \circ z_{\Omega^2})^*(N_{\mathcal{F}}) \cong {z_{\Omega^2}}^*\big({t_{\Omega^2}}^*(N_{\mathcal{F}})\big)$. Hence we have a pullback diagram

$$\begin{tikzcd} E \cong W \arrow[r, "pr_{{j_0}^*(N_{\mathcal{F}})}"] \arrow[d, "pr_{N_{\Omega^2}}"'] & {j_0}^*(N_{\mathcal{F}}) \arrow[r, hook] \arrow[d, "p"] & {t_{\Omega^2}}^*(N_{\mathcal{F}}) \arrow[d] \\ N_{\Omega^2} \arrow[r, "p_{\Omega^2}"'] & \Omega^2 \arrow[r, hook, "z_{\Omega^2}"'] & N_{\Omega^2}. \end{tikzcd} \tag{3.6}$$

Since $p_{\Omega^2} : N_{\Omega^2} \to \Omega^2$ is a vector bundle with zero section z_{Ω^2} the lower horizontal map $z_{\Omega^2} \circ p_{\Omega^2}$ in this diagram is homotopic to the identity. Using paracompactness of N_{Ω^2} (is it given?), we therefore finally get $E \cong W = (z_{\Omega^2} \circ p_{\Omega^2})^*\big({t_{\Omega^2}}^*(N_{\mathcal{F}})\big) \cong id^*\big({t_{\Omega^2}}^*(N_{\mathcal{F}})\big) \cong {t_{\Omega^2}}^*(N_{\mathcal{F}})$ (see [**Hus94**, Chapter 3, Theorem 4.7]). We thus have a vector bundle isomorphism $f : E \to {t_{\Omega^2}}^*(N_{\mathcal{F}})$ over N_{Ω^2}. This means that the upper triangle in the diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "pr_{N_{\Omega^2}}"] \arrow[ddr, bend right, "p_1"'] & & {t_{\Omega^2}}^*(N_{\mathcal{F}}) \arrow[dl] \arrow[ddl, bend left, "p_2"] \\ & N_{\Omega^2} \arrow[d, "p_{\Omega^2}"] & \\ & \Omega^2 & \end{tikzcd}$$

commutes. This shows that p_2 is a vector bundle projection since p_1 is one ([**Hus94**, Chapter 2, Corollary 6.4]).

We now come to the final step of the proof that

$$(j \times j)_! = \pm j_{0!} \circ i_{\mathcal{F}!}$$

holds: By (the proof of) proposition B.2 of [**GH09**] the diagram

$$\begin{tikzcd} H_i(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}) \arrow[r, "\cap \tau_{\mathcal{F}}", "\cong"'] \arrow[d] & H_{i-n}(N_{\mathcal{F}}) \arrow[r, "\cong"'] \arrow[d] & H_{i-n}(\mathcal{F}) \arrow[d] \\ H_i(N_{\mathcal{F}}, N_{\mathcal{F}} - \Omega^2) \arrow[r, "\cap \tau_{\mathcal{F}}", "\cong"'] & H_{i-n}(N_{\mathcal{F}}, N_{\mathcal{F}|\mathcal{F}-\Omega^2}) \arrow[r, "\cong"'] & H_{i-n}(\mathcal{F}, \mathcal{F} - \Omega^2) \end{tikzcd}$$

commutes and $\cap \tau_{\mathcal{F}} : H_i(N_{\mathcal{F}}, N_{\mathcal{F}} - \Omega^2) \to H_{i-n}(N_{\mathcal{F}}, N_{\mathcal{F}|\mathcal{F}-\Omega^2})$ is an isomorphism.
Let $U_{\Omega^2} := t_{\Omega^2}(N_{\Omega^2})$ denote the tubular neighbourhood of Ω^2 in $\mathcal{F}$ and let ${t_{\Omega^2}}^*(\tau_{\mathcal{F}}) \in$

$H^n(t_{\Omega^2}{}^*(N_{\mathcal{F}}), t_{\Omega^2}{}^*(N_{\mathcal{F}}) - N_{\Omega^2})$ be the Thom class induced by the class $\tau_{\mathcal{F}}$. $t_{\Omega^2}{}^*(\tau_{\mathcal{F}})$ pulls back to (a class corresponding to) $j_0{}^*(\tau_{\mathcal{F}}) \in H^n(j_0{}^*(N_{\mathcal{F}}), j_0{}^*(N_{\mathcal{F}}) - \Omega^2)$ as the diagram 3.6 shows. It follows that the following diagram commutes:

$$\begin{tikzcd}
H_i(\Lambda^2) \arrow[d] \arrow[ddrr, "i_{\mathcal{F}!}"] & & \\
H_i(\Lambda^2, \Lambda^2 - \mathcal{F}) \arrow[d, "\cong"'] & & \\
H_i(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}) \arrow[r, "\cap\tau_{\mathcal{F}}", "\cong"'] \arrow[d] & H_{i-n}(N_{\mathcal{F}}) \arrow[r, "\cong"'] \arrow[d] & H_{i-n}(\mathcal{F}) \arrow[d] \arrow[dddddd, bend left, "j_{0!}"] \\
H_i(N_{\mathcal{F}}, N_{\mathcal{F}} - \Omega^2) \arrow[r, "\cap\tau_{\mathcal{F}}", "\cong"'] \arrow[d, "\cong"', "\text{excision}"] & H_{i-n}(N_{\mathcal{F}}, N_{\mathcal{F}|\mathcal{F}-\Omega^2}) \arrow[r, "\cong"'] \arrow[d, "\cong"', "\text{excision}"] & H_{i-n}(\mathcal{F}, \mathcal{F} - \Omega^2) \arrow[d, "\cong"', "\text{excision}"] \\
H_i(N_{\mathcal{F}|U_{\Omega^2}}, N_{\mathcal{F}|U_{\Omega^2}} - \Omega^2) \arrow[r, "\cap\tau_{\mathcal{F}}", "\cong"'] \arrow[d, "\cong"'] & H_i(N_{\mathcal{F}|U_{\Omega^2}}, N_{\mathcal{F}|U_{\Omega^2}-\Omega^2}) \arrow[r, "\cong"'] \arrow[d, "\cong"'] & H_{i-n}(U_{\Omega^2}, U_{\Omega^2} - \Omega^2) \arrow[d, "\cong"'] \\
H_i(t_{\Omega^2}{}^*(N_{\mathcal{F}}), t_{\Omega^2}{}^*(N_{\mathcal{F}}) - \Omega^2) \arrow[r, "\cap t_{\Omega^2}{}^*(\tau_{\mathcal{F}})", "\cong"'] \arrow[d, "\cong"'] & H_{i-n}(t_{\Omega^2}{}^*(N_{\mathcal{F}}), t_{\Omega^2}{}^*(N_{\mathcal{F}})_{|N_{\Omega^2}-\Omega^2}) \arrow[r, "\cong"'] \arrow[d, "\cong"'] & H_{i-n}(N_{\Omega^2}, N_{\Omega^2} - \Omega^2) \arrow[d, "="'] \\
H_i(W, W - \Omega^2) \arrow[r, "\cap(pr_{j_0{}^*(N_{\mathcal{F}})})^*\left(j_0{}^*(\tau_{\mathcal{F}})\right)", "\cong"'] \arrow[dr, "\cap\tau_W"'] & H_{i-n}\big(W, W - j_0{}^*(N_{\mathcal{F}})\big) \arrow[r, "\cong"'] \arrow[d, "\cap(pr_{N_{\Omega^2}})^*(\tau_0)"] & H_{i-n}(N_{\Omega^2}, N_{\Omega^2} - \Omega^2) \arrow[d, "\cap\tau_0"] \\
 & H_{i-2n}(W) \arrow[r, "\cong"'] \arrow[dr, "\cong"'] & H_{i-2n}(N_{\Omega^2}) \arrow[d, "\cong"'] \\
 & & H_{i-2n}(\Omega^2).
\end{tikzcd}$$

Hence, if we choose $\tau_W = (pr_{N_{\Omega^2}})^*(\tau_0) \cup (pr_{j_0{}^*(N_{\mathcal{F}})})^*\big(j_0{}^*(\tau_{\mathcal{F}})\big)$ to be the Thom class of the normal bundle $W \cong E \to \Omega^2$, then the composition down the left and then along the bottom in the above diagram is $(j \times j)_!$.
This shows that the diagram

(3.7)
$$\begin{tikzcd}
 & H_{i-n}(\mathcal{F}) \arrow[dl, "j_{0!}"'] & \\
H_{i-2n}(\Omega \times \Omega) & & H_i(\Lambda \times \Lambda) \arrow[ll, "(j\times j)_!"'] \arrow[ul, "i_{\mathcal{F}!}"']
\end{tikzcd}$$

really is commutative. If instead we had chosen $\widetilde{\tau_W} := (pr_{j_0{}^*(N_{\mathcal{F}})})^*\big(j_0{}^*(\tau_{\mathcal{F}})\big) \cup (pr_{N_{\Omega^2}})^*(\tau_0)$, the diagram would only commute up a factor $(-1)^{n^2}$.

We seem to have a choice here if n is odd, the choice of setting

$$\tau_\Omega \times \tau_\Omega = \begin{cases} \tau_W, \text{ or} \\ \widetilde{\tau_W} = (-1)^{n^2}\tau_W. \end{cases}$$

We choose to equip E with the orientation $\widetilde{\tau_W}$ of $\widetilde{W} := N_{\Omega^2} \oplus j_0^*(N_{\mathcal{F}}) \cong j_0^*(N_{\mathcal{F}}) \oplus N_{\Omega^2} = W \cong E$. That is, we set

$$\tau_\Omega \times \tau_\Omega = (-1)^{n^2}\tau_W.$$

Note that this choice does not affect the order of the factors in $\Omega^2 = \Omega \times \Omega$:

$$\begin{tikzcd} E \arrow[r, "\cong"'] \arrow[drr] & W \arrow[rr, "\cong"'] \arrow[dr] & & \widetilde{W} \arrow[dl] \\ & & \Omega_1 \times \Omega_2 & \end{tikzcd}$$

where we have indexed the two copies of Ω to show that their order is not affected. For $a \in H_i(\Lambda)$ and $b \in H_j(\Lambda)$ we have $(a \cap \tau_\Omega) \times (b \cap \tau_\Omega) = (-1)^{nj}(a \times b) \cap (\tau_\Omega \times \tau_\Omega)$ ([**Bre93**, Chapter VI, Theorem 5.4]). Hence

$$\begin{aligned} j_!(a) \times j_!(b) &= (a \cap \tau_\Omega) \times (b \cap \tau_\Omega) = (-1)^{nj}(a \times b) \cap (\tau_\Omega \times \tau_\Omega) \\ &= (-1)^{nj}(-1)^{n^2}(a \times b) \cap \tau_W \\ &= (-1)^{n(n-j)}(j \times j)_!(a \times b). \end{aligned} \tag{3.8}$$

Note that the sign change here is exactly the sign-correction that was introduced in the definition of the Chas-Sullivan product.

There is another pullback square which shows that $\Omega \times \Omega$ is a codimension-n-submanifold of $\mathcal{F}$, namely

$$\begin{tikzcd} \Omega^2 \arrow[r, hook, "j_0"'] \arrow[d, "\phi"] & \mathcal{F} \arrow[d, "\phi"] \\ \Omega \arrow[r, hook, "j"] & \Lambda. \end{tikzcd}$$

It is stated in section 9.3 of [**GH09**] that $\phi : \Omega^2 \to \Omega$ is an embedding in the Hilbert manifold sense and that the square above is a pullback. From this the commutativity of the square

$$\begin{tikzcd} H_i(\mathcal{F}) \arrow[r, "\phi_*"] \arrow[d, "j_{0!}"'] & H_{i-n}(\Lambda) \arrow[d, "j!"] \\ H_{i-n}(\Omega \times \Omega) \arrow[r, "\phi_*"] & H_{i-n}(\Omega) \end{tikzcd} \tag{3.9}$$

follows.

Thus, (3.8), (3.7) and (3.9) all together yield the commutativity of

$$\begin{tikzcd}
H_i(\Lambda) \times H_j(\Lambda) \arrow[r, "\times"] \arrow[d, "(-1)^{n(n-j)} j_! \times j_!"] \arrow[rrr, bend left, "*'"] & H_{i+j}(\Lambda \times \Lambda) \arrow[r, "i_!"] \arrow[d, "(j \times j)_!"] & H_{i+j-n}(\mathcal{F}) \arrow[r, "\phi_*"] \arrow[d, "j_{0!}"'] & H_{i+j-n}(\Lambda) \arrow[d, "j!"] \\
H_{i-n}(\Omega) \times H_{j-n}(\Omega) \arrow[r, "\times"] \arrow[rrr, bend right, "\star"'] & H_{i+j-2n}(\Omega \times \Omega) \arrow[r, "="] & H_{i+j-2n}(\Omega \times \Omega) \arrow[r, "\phi_*"] & H_{i+j-2n}(\Omega).
\end{tikzcd}$$

Here the lower vertical composition $\star$ is the Pontrjagin product.
This shows that

$$\begin{aligned} j_!(a * b) &= j_!((-1)^{n(n-j)} a *' b) = (-1)^{n(n-j)}(-1)^{n(n-j)} j_!(a) \star j_!(b) \\ &= j_!(a) \star j_!(b). \end{aligned}$$

We have thus proved equation (3.1).

This completes the partial proof of the proposition. □

4. The Chas-Sullivan algebra of spheres

Let now again $M = S^n$. We have already seen that

$$H_i(\Lambda S^n) = H_i(S^n) \oplus \bigoplus_{r \in \mathbb{N}} H_i\big(\Lambda S^{n \le 2\pi^2 r^2}, \Lambda S^{n < 2\pi^2 r^2}\big) = H_i(S^n) \oplus \bigoplus_{r \in \mathbb{N}} H_{i-\lambda_r}(T^1 S^n).$$

with $\lambda_r = (2r-1)(n-1)$. Here $2\pi^2 r^2$ and λ_r are the critical levels and the Morse indices of the standard metric on S^n. Each critical submanifold B_r of positive energy $2\pi^2 r^2$ is diffeomorphic to $T^1 S^n$.

We plot this in an energy E versus degree d diagram: At the coordinate (E, d) we have the entry $H_d(\Lambda S^{n \le E}, \Lambda S^{n < E})$ which is zero if E is not a critical value. The only nonzero entires are $H_d\big(\Lambda S^{n \le 2\pi^2 r^2}, \Lambda S^{n < 2\pi^2 r^2}\big)$ for some d in the range $\lambda_r \le d \le \lambda_r + 2n - 1 = \lambda_r + dim(T^1 S^n)$. For any d we have that

$$H_d\big(\Lambda S^{n \le 2\pi^2 r^2}, \Lambda S^{n < 2\pi^2 r^2}\big) \cong H_d\big(\Lambda S^{n \le 2\pi^2 r^2}, \Lambda S^{n \le 2\pi^2 (r-1)^2}\big) \cong H_d(\Lambda S^n)$$

([**GH09**, Section 13 and Appendix A] or [**Zil77**]). The diagram thus shows from which critical level a global homology class originates. In other terms, what we see is the level homology, but since it survives to global homology, we actually see all the homology of ΛS^n. Here surviving means two things, the relative classes $H_d\big(\Lambda S^{n \le 2\pi^2 r^2}, \Lambda S^{n < 2\pi^2 r^2}\big)$ can be completed, i.e. $H_d\big(\Lambda S^{n \le 2\pi^2 r^2}\big) \cong H_d\big(\Lambda S^{n \le 2\pi^2 r^2}, \Lambda S^{n < 2\pi^2 r^2}\big) \oplus H_d\big(\Lambda S^{n < 2\pi^2 r^2}\big)$ and they survive, i.e. $H_d\big(\Lambda S^{n \le 2\pi^2 r^2}\big) \to H_d(\Lambda S^n)$ is injective. One could also see this as the first page of the spectral sequence given by the filtration $S^n \cong \Lambda S^{n \le 0} \subset \Lambda S^{n \le 2\pi^2} \subset \Lambda S^{n \le 2\pi^2 2^2} \subset \dots$ (compare [**HR13**, Section 2]). This sequence degenerates on the first page.

Thus, with integer coefficients, the full loop space homology $H_*(\Lambda S^n; \mathbb{Z})$ is given by

- for n odd

$\uparrow d$				
$6n-5$				$\mathbb{Z}$
$6n-6$				$\mathbb{Z}$
$5n-4$			$\mathbb{Z}$	
$5n-5$				$\mathbb{Z} = \langle \sigma_3 \rangle$
$4n-3$			$\mathbb{Z}$	
$4n-4$			$\mathbb{Z}$	
$3n-2$		$\mathbb{Z} = \langle \Theta \rangle$		
$3n-3$			$\mathbb{Z} = \langle \sigma_2 \rangle$	
$2n-1$		$\mathbb{Z} = \langle U \rangle$		
$2n-2$		$\mathbb{Z}$		
n	$\mathbb{Z} = \langle E \rangle$			
$n-1$		$\mathbb{Z} = \langle \sigma_1 \rangle$		
0	$\mathbb{Z} = \langle A \rangle$			
	0	$2\pi^2 1^2$	$2\pi^2 2^2$	$2\pi^2 3^2 \ \to E$

- and for even n

$\uparrow d$				
$6n-5$				0
$6n-6$				$\mathbb{Z}_2$
$5n-4$			$\mathbb{Z}$	
$5n-5$				$\mathbb{Z} = \langle \sigma_3 \rangle$
$4n-3$			0	
$4n-4$			$\mathbb{Z}_2$	
$3n-2$		$\mathbb{Z} = \langle \Theta \rangle$		
$3n-3$			$\mathbb{Z} = \langle \sigma_2 \rangle$	
$2n-1$		0		
$2n-2$		$\mathbb{Z}_2$		
n	$\mathbb{Z} = \langle E \rangle$			
$n-1$		$\mathbb{Z} = \langle \sigma_1 \rangle$		
0	$\mathbb{Z} = \langle A \rangle$			
	0	$2\pi^2$	$8\pi^2$	$18\pi^2 \ \to E$

We have also indicated some important generators, namely

- $A \in H_0(\Lambda S^n; \mathbb{Z})$ which corresponds to the generator of $H_0(S^n; \mathbb{Z})$. It generates the intersection algebra of S^n: $A \bullet A = 0$.

- $\sigma_1 \in H_{n-1}(\Lambda S^n; \mathbb{Z})$ is the generator that corresponds to the Kronecker dual of the Thom class τ_1 of the bundle $\Gamma_1^- \to B_1$. More generally we choose $\tau_r(\sigma_r) = 1$ for all $r \geq 1$.

- $E \in H_n(\Lambda S^n; \mathbb{Z})$ which corresponds to the orientation class of S^n. It is thus the unit of the intersection algebra $\big(H_{*+n}(S^n; \mathbb{Z}), \bullet\big) \cong \bigwedge(a)$ on S^n. Here a has degree $-n$ and is sent to A under this isomorphism.

- $U \in H_{2n-1}(\Lambda S^n; \mathbb{Z})$ exists only for odd n and is sent via $j_!$ to x which generates the Pontrjagin algebra $\big(H_*(\Omega S^n; \mathbb{Z}), \star\big) \cong \mathbb{Z}[x]$ with $|x| = n-1$.

- $\Theta \in H_{3n-2}(\Lambda S^n;\mathbb{Z})$ which corresponds to the orientation class of T_1S^n under the Thom isomorphism of the negative normal bundle of the first (other than S^n) critical submanifold B_1.

We now look at the Chas-Sullivan structure on the above graded modules. We have ([**CJY04**, Theorem 2])

- for $n \geq 3$ and n odd
$$\big(H_{*+n}(\Lambda S^n;\mathbb{Z}), *\big) \cong \bigwedge(a) \otimes \mathbb{Z}[x]$$
with $|a| = -n$ and $|x| = n-1$ on the right-hand side

- and for $n \geq 2$ and n even
$$\big(H_{*+n}(\Lambda S^n;\mathbb{Z}), *\big) \cong \big(\bigwedge(s_1) \otimes \mathbb{Z}[a,t]\big)/(a^2, s_1 \otimes a, 2a \otimes t)$$
with $|a| = -n$, $|s_1| = -1$, $|t| = 2n-2$ on the right-hand side.

The elements on the left-hand side are all shifted upward in degree by n, so that the generators on the right correspond to the generators of the free loop space homology indicated above:
$$A \leftrightarrow a,\ \sigma_1 \leftrightarrow s_1,\ U \leftrightarrow x,\ \Theta \leftrightarrow t.$$
It is important to use the sign correction as in [**GH09**] for the definition of $*$ here. Otherwise the commutativity behaviour on the two sides to the algebra homomorphisms differ.
For us it is important to know that

- for $n \geq 3$ and n odd each homology class (except for E) can be written as a (Chas-Sullivan) product of A and U.

- and for $n \geq 4$ and n even each homology class (except for E) can be written as a (Chas-Sullivan) product of A, σ_1 and Θ.

- the class E is a neutral element for the Chas-Sullivan product for n even and odd.

Moreover, for any n there is a nonnilpotent homology class $\Theta \in H_{3n-2}(\Lambda S^n;\mathbb{Z})$:
$$\Theta^k = \Theta^{*k} \neq 0,\ \forall k \in \mathbb{N}.$$
In the odd case, $\Theta = U * U = U^{*2}$. Furthermore, multiplication with this class is an isomorphism
$$*\Theta : H_k(\Lambda S^n;\mathbb{Z}) \xrightarrow{\cong} H_{k+3n-2-n}(\Lambda S^n;\mathbb{Z})$$
for all $k > 0$. The relation
$$\sigma_r = \sigma_1 * \Theta^{*(r-1)},$$
valid for odd and even n, will be used later.
Note that multiplication with Θ corresponds to "jumping" from one critical level of the standard metric to the next: induced by multiplication with Θ we have
$$H_i\big(\Lambda S^{n\leq 2\pi^2r^2}, \Lambda S^{n<2\pi^2r^2}\big) \xrightarrow{\cong} H_{i+2n-2}\big(\Lambda S^{n\leq 2\pi^2(r+1)^2}, \Lambda S^{n<2\pi^2(r+1)^2}\big)$$
for any r (see [**GH09**, Theorem 13.4 (3)]).
All of this (and more) is summarized in Lemma 5.4 of [**HR13**], proofs are given in [**GH09**, Section 13]. We give a diagrammatic summary of the situation:

- for n odd we have

$\uparrow d$				
$6n-5$				U^{*5}
$6n-6$				$A * U^{*6}$
$5n-4$			$U^{*4} = \Theta^{*2}$	
$5n-5$				$A * U^{*5} = \sigma_1 * \Theta^{*2} = \sigma_3$
$4n-3$			$U^{*3} = U * \Theta$	
$4n-4$			$A * U^{*4} = A * \Theta^{*2}$	
$3n-2$		$U^{*2} := \Theta$		
$3n-3$			$A * U^{*3} = \sigma_1 * \Theta = \sigma_2$	
$2n-1$		U		
$2n-2$		$A * U^{*2} = A * \Theta$		
n	E			
$n-1$		$A * U = \sigma_1$		
0	A			
	0	$2\pi^2$	$8\pi^2$	$18\pi^2 \ \rightarrow E$

- and for even n we get

$$
\begin{array}{c|cccc}
\uparrow d & & & & \\
6n-5 & & & & 0 \\
6n-6 & & & & A * \Theta^{*3} \\
5n-4 & & & \Theta^{*2} & \\
5n-5 & & & & \sigma_1 * \Theta^{*2} := \sigma_3 \\
4n-3 & & & 0 & \\
4n-4 & & & A * \Theta^{*2} & \\
3n-2 & & \Theta & & \\
3n-3 & & & \sigma_1 * \Theta := \sigma_2 & \\
2n-1 & & 0 & & \\
2n-2 & & A * \Theta & & \\
n & E & & & \\
n-1 & & \sigma_1 & & \\
0 & A & & & \\
\hline
 & 0 & 2\pi^2 & 8\pi^2 & 18\pi^2 \ \to E
\end{array}
$$

CHAPTER 3

$\mathbb{Z}_2$-equivariant products

In this chapter we do the following:

(1) In the first section we introduce two actions of $\mathbb{Z}_2$ on Λ. These actions are named θ and ϑ. ϑ is the orientation-reversal of loops and θ is orientation-reversal plus a shift of the starting point.
(2) In the second section we introduce a product P_G on the homology of Λ/G for any finite group G acting on Λ. It is defined using the Chas-Sullivan product and the transfer maps $tr : H_i(\Lambda/G) \to H_i(\Lambda)$, which can be associated to the ramified covering map $\Lambda \to \Lambda/G$.
It turns out that $H_*(\Lambda/\theta) \cong H_*(\Lambda/\vartheta)$ and that $P_\theta \cong P_\vartheta$.
(3) In section 3 we define a tubular neighbourhood of $\mathcal{F}/(\vartheta \times \vartheta)$ inside $\Lambda^2/(\vartheta \times \vartheta)$ and use it to define a product $A_\theta : H_i(\Lambda/\theta) \times H_j(\Lambda/\theta) \to H_{i+j-n}(\Lambda/\theta)$. It is defined in the spirit of the Chas-Sullivan product and it turns out that $A_\theta = P_\theta$.
(4) In the fourth section we try to do the same as in the third section but now for the action ϑ. We construct a neighbourhood of $\mathcal{F}/\psi$ inside Λ^2/ψ that is the image of the quotient of a vector bundle. Here ψ is an appropriate $\mathbb{Z}_2$-action. The product A_ϑ does not always coincide with P_θ.
(5) The rest of the chapter is devoted to other products, for example an equivariant one that should be compared to $P_{\mathbb{Z}_2}$.

1. The two main $\mathbb{Z}_2$-actions θ and ϑ

We are going to consider mainly two actions, called θ and ϑ, of the abelian group $\mathbb{Z}_2$ on Λ. The actions will be induced by automorphisms of the circle S^1, the domain of loops in M. We thus start with a note on the topology of S^1: We consider the Lie groups

- $(\mathbb{R}, +)$ with the Euclidean topology and standard smooth structure.
- $\mathbb{C}^* = (\mathbb{R}^2 \setminus \{0\}, \cdot)$ with the Euclidean topology and the complex multiplication and standard smooth structure.
- $S^1 \subset \mathbb{C}^*$ a Lie subgroup of $\mathbb{C}^*$.

The subgroup $(\mathbb{Z}, +) \subset (\mathbb{R}, +)$ acts on $\mathbb{R}$ via $(n, r) \mapsto n + r$. Let $q : \mathbb{R} \mapsto \mathbb{R}/\mathbb{Z}$ denote the (smooth) quotient map.
Consider also the smooth surjection $e : \mathbb{R} \to S^1,\ t \mapsto e^{2\pi i t}$. It is an open map, hence also a quotient map. We have

$$\begin{tikzcd} & \mathbb{R} \arrow[dl, "q"'] \arrow[dr, "e"] & \\ \mathbb{R}/\mathbb{Z} & & S^1 \end{tikzcd}$$

As $q(x) = q(y) \Leftrightarrow x - y \in \mathbb{Z} \Leftrightarrow 1 = e^{2\pi i(x-y)} \Leftrightarrow e(x) = e(y)$, the map $\mathbb{R}/\mathbb{Z} \to S^1,\ [t] \mapsto e^{2\pi i t}$ is a homeomorphism. In particular, $\mathbb{R}/\mathbb{Z}$ is compact and Hausdorff. The surjective restrictions $e_{|[0,1]} : [0,1] \mapsto S^1$ and $q_{|[0,1]} : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ factor through the quotient $[0,1]/\{0,1\}$:

$$\begin{tikzcd}
& {[0,1]} \arrow[dl, "q_{|[0,1]}"'] \arrow[d] \arrow[dr, "e_{|[0,1]}"] & \\
\mathbb{R}/\mathbb{Z} & {[0,1]/\{0,1\}} \arrow[l, "\cong", "\tilde{q}"'] \arrow[r, "\cong"', "\tilde{e}"] & S^1
\end{tikzcd}$$

Since $e_{|[0,1]}$ and $q_{|[0,1]}$ are continuous surjections between compact Hausdorff spaces, they are quotient maps. Since they make the same identifications as $[0,1] \to [0,1]/\{0,1\}$, it follows that $\tilde{e}$ and $\tilde{q}$ are homeomorphisms.

Given a continuous map $\gamma : S^1 \to M$, the map $\gamma' := \gamma \circ e : \mathbb{R} \to M$ is 1-periodic (and continuous): $\gamma'(t+1) = \gamma\big(e(t+1)\big) = \gamma\big(e(t)\big) = \gamma'(t)$. Conversely, if a map $\gamma' : \mathbb{R} \to M$ is 1-periodic, then it factors through S^1: $x - y = n \in \mathbb{Z}$ implies $\gamma'(x) = \gamma'(y+n) = \cdots = \gamma'(y)$. So, there is a unique continuous map $\gamma : S^1 \to M$ with $\gamma' = \gamma \circ e$. Hence there is a bijection between $LM = C^0(S^1, M)$ and $\{\gamma' \in C^0(\mathbb{R}, M) |\ \gamma' \text{ is 1-periodic}\}$. Thus, we usually write t for elements of S^1 and $\gamma(t)$ for $\gamma : S^1 \to M$, and treat γ as a 1-periodic map $\mathbb{R} \to M$.

Let us now describe the two $\mathbb{Z}_2$-actions θ and ϑ:

(1) The action θ: It is generated by the involution $\gamma \mapsto \frac{1}{2}.\bar{\gamma}$, where $\bar{\gamma}(t) := \gamma(1-t)$ and $\frac{1}{2}.\gamma(t) = \gamma(t + \frac{1}{2})$. So $\frac{1}{2}.\bar{\gamma}(t) = \gamma(1 - t + \frac{1}{2})$. That is, we move the starting point of the curve and in addition reverse its orientation. We use the same name for this involution, so

$$\theta : \Lambda \to \Lambda,\ \theta(\gamma) := \frac{1}{2}.\bar{\gamma}.$$

The involution θ is induced by an automorphism of S^1: We view $S^1 \subset \mathbb{C}$. Let $k : S^1 \to S^1$ be the involution given by complex conjugation and let, for $t \in [0,1]$, $f_t : S^1 \to S^1,\ z \mapsto e^{2\pi i t} \cdot z$. These maps are continuous. $f_{\frac{1}{2}}$ is multiplication with $e^{\pi i} = -1$. The map

$$f_{\frac{1}{2}} \circ k : S^1 \to S^1$$

is again an automorphism of S^1. It induces the involution θ on mapping spaces:

$$\theta = (f_{\frac{1}{2}} \circ k)^\sharp : C^0(S^1, M) \to C^0(S^1, M);\ \gamma \mapsto \gamma \circ f_{\frac{1}{2}} \circ k,$$

since

$$\begin{tikzcd}
t \arrow[r, mapsto] \arrow[d, mapsto, "e"'] & 1-t \arrow[d, mapsto, "e"] \\
e^{2\pi i t} \arrow[r, mapsto, "k"'] & e^{-2\pi i t}
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[r, mapsto] \arrow[d, mapsto, "e"'] & t + \frac{1}{2} \arrow[d, mapsto, "e"] \\
e^{2\pi i t} \arrow[r, mapsto, "f_{\frac{1}{2}}"'] & -e^{2\pi i t}
\end{tikzcd}$$

Note that since k is $\mathbb{R}$-linear, we have $f_{\frac{1}{2}} \circ k = k \circ f_{\frac{1}{2}}$ and $\theta = (k \circ f_{\frac{1}{2}})^\sharp$.

(2) The action ϑ: This action is generated simply by the involution $\gamma \mapsto \bar{\gamma}$, so this is just reversing the orientation of loops. We again also denote the involution itself by ϑ. We obviously have

$$\vartheta = k^\sharp : C^0(S^1, M) \to C^0(S^1, M);\ \gamma \mapsto \gamma \circ k.$$

Let us introduce the notation $\chi_t := (f_t)^\sharp : C^0(S^1, M) \to C^0(S^1, M)$. Note that since $\sharp$ is (contravariantly) functorial we have

$$\theta = k^\sharp \circ (f_{\frac{1}{2}})^\sharp = \vartheta \circ \chi_{\frac{1}{2}} = \vartheta \circ \frac{1}{2}. = \frac{1}{2}. \circ \vartheta,$$

and

$$\vartheta = \theta \circ \frac{1}{2}. = \frac{1}{2}. \circ \theta. \tag{1.1}$$

The two actions θ and ϑ are continuous actions of $\mathbb{Z}_2$ on ΛM: This follows since the inclusion $\Lambda = \Lambda M = H^1(S^1, M) \hookrightarrow C^0(S^1, M) = LM$ is continuous ([**Kli82**, Lemma 2.4.6] or [**Kli78**, Theorem 1.2.10]) and $\mathbb{Z}_2$ is discrete. More explicitly, it follows from Lemma 2.2.1 and Theorem 2.2.5 in [**Kli78**] as it is shown there that θ and ϑ are isometric involutions and since $\mathbb{Z}_2 \cong O(1) \subset O(2)$ carries the discrete topology. We also remark that χ_t are isometries.

An immediate consequence of equation (1.1) is that

$$\vartheta_* = \theta_* : H_i(\Lambda) \to H_i(\Lambda)$$

as $\chi_{\frac{1}{2}} = \frac{1}{2}.$ is homotopic to the identity.

Let Λ/θ and Λ/ϑ be the quotients under the actions θ and ϑ respectively. We have

Proposition 1.1. *The singular homology groups $H_*(\Lambda/\theta; R)$ and $H_*(\Lambda/\vartheta; R)$ with arbitrary coefficients R are isomorphic.*

Proof. This is quite obvious if we write ϑ as $\vartheta \circ id$. Then

$$\vartheta = \vartheta \circ id = \vartheta \circ \chi_{\frac{1}{2}} \circ \chi_{\frac{1}{2}} = \theta \circ \chi_{\frac{1}{2}} = \theta \circ \chi_{\frac{1}{4}} \circ \chi_{\frac{1}{4}}$$

and hence

$$\chi_{\frac{1}{4}} \circ \vartheta = \vartheta \circ (\chi_{\frac{1}{4}})^{-1} = \theta \circ \chi_{\frac{1}{4}}$$

and thus $\chi_{\frac{1}{4}}$ is an equivariant with respect to ϑ and θ. We get a continuous map

$$\chi_{\frac{1}{4}}/\mathbb{Z}_2 : \Lambda/\theta \to \Lambda/\vartheta.$$

Let us check that the inverse $\chi_{\frac{3}{4}} = \chi_{-\frac{1}{4}}$ of $\chi_{\frac{1}{4}}$ is also equivariant:

$$\begin{tikzcd}
\gamma(t - \frac{1}{4}) \arrow[d, mapsto, "\vartheta"'] & \gamma(t) \arrow[l, mapsto, "-\frac{1}{4}."'] \arrow[d, mapsto, "\theta"] \\
\gamma(1 - t + \frac{1}{4}) & \gamma(1 - t + \frac{1}{2}) \arrow[l, mapsto, "-\frac{1}{4}."]
\end{tikzcd}$$

obviously commutes, so we also get a continuous map $\chi_{-\frac{1}{4}}/\mathbb{Z}_2 : \Lambda/\vartheta \to \Lambda/\theta$ and $\chi_{\frac{1}{4}}/\mathbb{Z}_2 \circ \chi_{-\frac{1}{4}}/\mathbb{Z}_2 = id_{\Lambda/\theta}$ and $\chi_{-\frac{1}{4}}/\mathbb{Z}_2 \circ \chi_{-\frac{1}{4}}/\mathbb{Z}_2 = id_{\Lambda/\vartheta}$. It follows that

$$\begin{tikzcd} H_i(\Lambda/\vartheta;R) \arrow[r, shift left, "(\chi_{\frac{1}{4}}/\mathbb{Z}_2)_*"] \arrow[r, phantom, "\cong"] & H_i(\Lambda/\theta;R) \arrow[l, shift left, "(\chi_{-\frac{1}{4}}/\mathbb{Z}_2)_*"] \end{tikzcd}$$

for all i. □

2. The transfer and the transfer product

Let G be a finite and discrete topological group acting continuously on $\Lambda = \Lambda M$, where M is a compact, connected, smooth manifold of dimension n.

If the action is free, then the map $q : \Lambda \to \Lambda/G$ is a covering map. This holds since Λ is Hausdorff: As G is compact and Hausdorff, it operates properly on Λ ([**LS15**, Satz 5.7]). Since G acts freely, $\Lambda \to \Lambda/G$ is even a principal G-bundle ([**LS15**, Folgerung 9.4]) and a normal covering ([**LS15**, Satz 9.3] and [**Hat02**, Proposition 1.40]). (If in addition M is simply-connected (i.e. $\pi_0(M) = 0, \pi_1(M) = 0$), then even $G \cong \pi_1(\Lambda/G) \cong$ the group of deck transformations of $\Lambda \to \Lambda/G$ since Λ is a connected manifold in that case ([**Kli78**, Corollary 2.1.5]). In fact, for $k \geq 1$, Λ is k-connected if and only if M is $(k+1)$-connected.)
If $q : \Lambda \to \Lambda/G$ is a covering map, we not only have a homomorphism $q_* : H_i(\Lambda;R) \to H_i(\Lambda/G;R)$ but also one in the other direction $tr : H_i(\Lambda/G;R) \to H_i(\Lambda;R)$ for singular homology with any coefficient group R. This homomorphism is called transfer and it is induced by a chain map which can be defined in very straight foreward and natural way on the singular chain complexes: just send a singular simplex of X/G to the sum of its $|G|$ distinct lifts. This it is explained in section 3.G of [**Hat02**].

Unfortunately no subgroup $G \subset O(2)$ (except the trivial) acts freely on Λ and thus the quotient maps $\Lambda \to \Lambda/\vartheta$ and $\Lambda \to \Lambda/\theta$ are not a covering maps. Nevertheless, also for nonfree actions of a finite group G on Λ we get transfer homomorphisms. They can be constructed in (at least) two ways:

(1) Any orbit map $q : X \to X/G$ of a continuous action of a finite discrete group G on a topological space X is a ramified covering map in the sense of [**Smi83**]. A d-fold ramified covering map is a continuous surjective finite-to-one map $p : X \to Y$ for which a continuous map $t_p : Y \to SP^d(X)$ exists. t_p maps a $y \in Y$ to the unordered d-tupel of points of $p^{-1}(y)$ counted with multiplicities. In [**Smi83**], a transfer homomorphism $tr : \widetilde{H}_i(X/G;R) \to \widetilde{H}_i(X;R)$ on reduced singular homology for $R = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_k$ is developed if X and X/G have the homotopy type of connected CW-complexes. It uses the homotopy-theoretic definition of homology. We will say a few word about the definition of transfer maps for ramified coverings at the end of this section. We extend the the transfer to unreduced singular homology of Λ by choosing a constant loop c_o as a base point. Since the action of G is trivial on the submanifold Λ^0 of constant loops, we define the transfer to be multiplication by $|G|$ on the additional $R \cong H_0(\{c_0\})$:

$$tr_0 := |G|(q_*)^{-1} : H_0(\{q(c_0)\}) \to H_0(\{c_0\}).$$

Alternatively we can refer to [**Dol86**], where a definition of a transfer for ramified coverings as a singular chain map is given. This only requires that X is Hausdorff and has the homotopy type of CW-complex.

(2) If X and X/G both have the homotopy type of CW-complexes and X is paracompact and Hausdorff, then by Theorem 1 of [**Mar59**] or Corollary 10.4 of [**Bre68**] Čech homology and singular homology of X and X/G, respectively, are isomorphic and we can apply Theorem 7.2 of Section 7 of Chapter 3 of [**Bre72**] to get a transfer $tr : H_i(X/G; R) \to H_i(X; R)$ for singular homology with arbitrary coefficients. Since Λ is a manifold, it is paracompact and Hausdorff and is has the homotopy type of a countable CW-complex by Corollary 2 of [**Mil59**]. The same also holds for Λ/G for any finite subgroup G of $O(2)$ acting as explained in the last section: In fact, Λ/G is paracompact, Hausdorff since $q : \Lambda \to \Lambda/G$ is a perfect map and Λ/G is locally contractible since the normal bundle to any orbit is an equivariant vector bundle ([**Bre72**, Chapter 1, Theorem 3.1], [**Eng89**, Theorem 4.4.15], [**Hin84**, Section 1.5]). Moreover, since Λ has the $O(2)$-homotopy type of $O(2)$-CW-complex, Λ/G has the homotopy type of countable CW-complex ([**Rad89**, Theorem 4.2], [**Ill90**, Theorem A], [**tD87**, Chapter II, Proposition 1.16], [**FP90**, Corollary 5.2.6]). See also Lemma A.4 in Appendix A of [**GH09**] for a comparison of Čech and singular homology groups of Λ.

These transfer homomorphisms have similar properties as the transfer of actual coverings. We now list of some of their properties and we are going to use them later:

(1) $(q_* \circ tr)(a) = |G|a$
([**Smi83**, Proposition 2.2], [**Bre72**, Chapter 3, (7.1)]).
(2) $(tr \circ q_*)(x) = \sum_{g\in G} g_*(x)$
([**Smi83**, Proposition 2.4], [**Bre72**, Chapter 3, (7.1)]).
(3) $q_* : H_i(X; \mathbb{Q})^G \to H_i(X/G; \mathbb{Q})$ is an isomorphism
([**Smi83**, Proposition 2.5], [**Bre72**, Chapter 3, Theorem 7.2]).

Here $H_i(X; \mathbb{Q})^G = \{x \in H_i(X; \mathbb{Q})| \; g_*(x) = x \text{ for all } g \in G\}$.
In what follows, we are only using the existence and the properties (1), (2), (3) of transfer homomorphisms.

Using the transfer we can define a homology product on Λ/G for any finite subgroup G of $O(2)$:

Definition 2.1. Let M be a compact, connected, oriented manifold. The "transfer product"

$$P_G : H_i(\Lambda/G; R) \times H_j(\Lambda/G; R) \to H_{i+j-n}(\Lambda/G; R)$$

on singular homology with $R = \mathbb{Z}$ or $R = \mathbb{Q}$ is defined by

$$P_G(a, b) := q_*\big(tr(a) * tr(b)\big) \tag{2.1}$$

for classes $a, b \in H_*(\Lambda/G; R)$, where $*$ is the Chas-Sullivan product.

Unraveling the definition, P_G is the composition

$$\begin{tikzcd}
H_i(\Lambda/G;R)\times H_j(\Lambda/G;R) \arrow[d, "\text{transfer}\times\text{transfer}"'] \\
H_i(\Lambda;R)\times H_j(\Lambda;R) \arrow[d, "\text{Chas-Sullivan product}"'] \\
H_{i+j-n}(\Lambda;R) \arrow[d, "q_*"'] \\
H_{i+j-n}(\Lambda/G;R)
\end{tikzcd}$$

which should be compared to the equivariant product of Section 5.
We have assumed M to be oriented since we are only going to compute the homology with rational coefficients later on.

We will mainly be interested in the case $G = \mathbb{Z}_2$. If $f : \Lambda \to \Lambda$ is the involution that generates a given $\mathbb{Z}_2$-action, we will write Λ/f for the quotient by the group action:

$$\Lambda/f := q_f(\Lambda) = \Lambda/\mathbb{Z}_2,$$

where q_f denotes the quotient map. The associated transfer and transfer product will be denoted by tr_f and P_f respectively.

Proposition 2.2. *The transfer product is associative if we assume that*

- $G = \mathbb{Z}_2$,
- *the involution generating the action induces a Chas-Sullivan algebra homomorphism.*

The commutativity properties are obviously the same as those of the Chas-Sullivan product.

For example, the above assumptions are satisfied when $f = \vartheta$ or $f = \theta$ (see Remark 3.7 in Chapter 4).

Proof. Let $f : \Lambda \to \Lambda$ be the involution that generates a given $\mathbb{Z}_2$-action. With the assumption

$$f_*(x * y) = f_*(x) * f_*(y)$$

we get (abbreviating $tr = tr_f$ here)

$$\begin{aligned}
P_f\big(P_f(a,b),c\big) &= q_*\Big(tr\big(P_f(a,b)\big) * tr(c)\Big) = q_*\Big(tr\big(q_*\big(tr(a) * tr(b)\big)\big) * tr(c)\Big) \\
&= q_*\bigg(\Big(tr(a) * tr(b) + f_*\big(tr(a) * tr(b)\big)\Big) * tr(c)\bigg) \text{ by property 2} \\
&= q_*\bigg(\Big(tr(a) * tr(b) + f_*\big(tr(a)\big) * f_*\big(tr(b)\big)\Big) * tr(c)\bigg) \\
&= q_*\Big(tr(a) * tr(b) * tr(c) + f_*\big(tr(a)\big) * f_*\big(tr(b)\big) * tr(c)\Big)
\end{aligned}$$

and

$$P_f\big(a, P_f(b,c)\big) = q_*\Big(tr(a) * \Big(tr(b) * tr(c) + f_*\big(tr(b) * tr(c)\big)\Big)\Big)$$
$$= q_*\Big(tr(a) * tr(b) * tr(c) + tr(a) * f_*\big(tr(b)\big) * f_*\big(tr(c)\big)\Big).$$

Property (2) implies that

$$(tr \circ q_*)\big(tr(a)\big) = tr(a) + f_*\big(tr(a)\big)$$

and Property (1) that

$$2tr(a) = tr\big(q_* \circ tr(a)\big).$$

Together that is

$$2tr(a) = tr(a) + f_*\big(tr(a)\big),$$

which shows that

$$tr(a) = f_*\big(tr(a)\big)$$

and hence

$$im(tr) \subset \{x \in H_*(\Lambda) |\ f_*(x) = x\}.$$

Therefore the above expressions $P_f\big(P_f(a,b),c\big)$ and $P_f\big(a, P_f(b,c)\big)$ are equal:

$$P_f\big(P_f(a,b),c\big) = 2q_*\big(tr(a) * tr(b) * tr(c)\big) = P_f\big(a, P_f(b,c)\big).$$

□

REMARK 2.3.

- An element $x \in H_*(\Lambda;\mathbb{Z})$ which does not pass to the quotient has the property of being sent to its additive inverse under f_*, i.e.

$$ker(q_*) \subset \{x \in H_*(\Lambda;\mathbb{Z}) |\ f_*(x) = -x\}.$$

  This holds since $x \in ker(q_*) \Leftrightarrow 0 = q_*(x) \Rightarrow 0 = (tr \circ q_*)(x) = x + f_*(x)$ by Property (2).
- Property (1) implies that the kernel of tr consists of torsion elements of order $2 = |\mathbb{Z}_2|$:

$$ker(tr) \subset \{a \in H_*(\Lambda/\mathbb{Z}_2;\mathbb{Z}) |\ 2a = 0\}$$

  since $tr(a) = 0 \Rightarrow |\mathbb{Z}_2|a = q_*\big(tr(a)\big) = 0$.

We now take homology with rational coefficients and make use of property (3):

PROPOSITION 2.4. *Let $f : \Lambda \to \Lambda$ be an involution which is trivial on the point curves (= constant loops). Then the pair $\big(H_*(\Lambda/f;\mathbb{Q}), P_f\big)$ is a unital algebra.*

PROOF. We only need to proof that there is a unit element e. Since we take homology with coefficients in $\mathbb{Q}$, the quotient map q_f induces a surjective homomorphism $q_{f_*} : H_i(\Lambda;\mathbb{Q}) \to H_i(\Lambda/f;\mathbb{Q})$. Hence for an arbitrary class $a \in H_*(\Lambda/f;\mathbb{Q})$ there is a class $x \in H_*(\Lambda;\mathbb{Q})$ with $q_{f_*}(x) = a$. Moreover $f_*(x) = x$. Since M is a compact, connected, oriented manifold of dimension n, there is a unit $E \in H_n(\Lambda;\mathbb{Q})$ for the Chas-Sullivan product. We define

$$e := \frac{1}{4} q_{f_*}(E).$$

This in nonzero, since E is the orientation class of the base manifold M, which is equivariantly embedded into ΛM via the point curves, and so $f_*(E) = E$ by assumption on f. We compute, using the properties of the transfer, that

$$\begin{aligned} P_f(a,e) &= \frac{1}{4} q_{f*}\big((tr_f \circ q_{f*})(x) * (tr_f \circ q_{f*})(E)\big) \\ &= \frac{1}{4} q_{f*}\big(x * E + x * f_*(E) + f_*(x) * E + f_*(x) * f*(E)\big) \\ &= \frac{1}{4} q_{f*}(4x * E) = q_{f*}(x) = a. \end{aligned}$$

Likewise $P_f(e,a) = a$. □

We thus have proved:

THEOREM 2.5. *Let M be a compact, connected, oriented manifold and $\Lambda = \Lambda M$ its free loop space. Let $f : \Lambda \to \Lambda$ be a continuous involution of ΛM which is the identity on point curves. Then $\big(H_*(\Lambda/f;\mathbb{Q}), P_f\big)$ is a graded-commutative unital $\mathbb{Q}$-algebra.*
If in addition the conditions of Proposition 2.2 are met, $\big(H_*(\Lambda/f;\mathbb{Q}), P_f\big)$ is associative, graded-commutative and unital.

□

In particular, for $n \geq 3$ the algebras $\big(H_*(\Lambda S^n/\vartheta;\mathbb{Q}), P_\vartheta\big)$ and $\big(H_*(\Lambda S^n/\theta;\mathbb{Q}), P_\theta\big)$ are associative, graded-commutative and unital.

PROPOSITION 2.6. *In the situation of the above theorem, if f in addition leaves base points fixed, i.e. $f(\gamma)(0) = \gamma(0)$ for all $\gamma \in \Lambda M$, then*

$$(ev_0/f)_* : \big(H_*(\Lambda/f;\mathbb{Q}), P_f\big) \to \big(H_*(M;\mathbb{Q}), \bullet\big)$$

is an algebra homomorphism up to scaling.

PROOF. $f(\gamma)(0) = \gamma(0)$ for all $\gamma \in \Lambda M$ means that $ev_0 : \Lambda M \to M$ is equivariant with respcet to the trivial $\mathbb{Z}_2$-action on the base M. Hence, using the transfer properties, for $a, b \in H_*(\Lambda/f;\mathbb{Q})$ with $q_{f*}(x) = a$ and $q_{f*}(y) = b$ and $x, y \in H_*(\Lambda;\mathbb{Q})^f := \{x \in H_*(\Lambda;\mathbb{Q}) |\, f_*(x) = x\}$, we get

$$\begin{aligned} (ev_0/f)_*\big(P_f(a,b)\big) &= \big((ev_0/f)_* \circ q_{f*}\big)\big(tr_f(a) * tr_f(b)\big) = (ev_0)_*\big(tr_f(a) * tr_f(b)\big) \\ &= (ev_0)_*\big(tr_f(a)\big) \bullet (ev_0)_*\big(tr_f(b)\big) = (ev_0)_*(2x) \bullet (ev_0)_*(2y) \\ &= \big((ev_0/f)_* \circ q_{f*}\big)(2x) \bullet \big((ev_0/f)_* \circ q_{f*}\big)(2y) \\ &= 4(ev_0/f)_*(a) \bullet (ev_0/f)_*(b). \end{aligned}$$

□

Note that ϑ preserves base points while θ does not.

Finally, for rational coefficients, there is a relation between the products P_ϑ and P_θ:

THEOREM 2.7. *The algebras $\big(H_*(\Lambda/\vartheta;\mathbb{Q}), P_\vartheta\big)$ and $\big(H_*(\Lambda/\theta;\mathbb{Q}), P_\theta\big)$ are isomorphic:*

$$\big(H_*(\Lambda/\vartheta;\mathbb{Q}), P_\vartheta\big) \cong \big(H_*(\Lambda/\theta;\mathbb{Q}), P_\theta\big).$$

PROOF. Let $a, b \in H_*(\Lambda/\vartheta)$. We define

$$\chi := \chi_{\frac{1}{4}}/\mathbb{Z}_2 : \Lambda/\vartheta \cong \Lambda/\theta$$

to be the homeomorphism induced by the equivariant homeomorphism $\chi_{\frac{1}{4}} : \Lambda \to \Lambda$ (see Section 1). If q_θ, q_ϑ denote the orbit maps of the two actions, the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\chi_{\frac{1}{4}}", "\cong"'] \arrow[d, "q_\vartheta"'] & \Lambda \arrow[d, "q_\theta"] \\ \Lambda/\vartheta \arrow[r, "\chi", "\cong"'] & \Lambda/\theta \end{tikzcd}$$

commutes. It follows that $\chi_*(a), \chi_*(b) \in H_*(\Lambda/\theta)$ and that we have

$$P_\theta\big(\chi_*(a), \chi_*(b)\big) = q_{\theta*}\Big(tr_\theta\big(\chi_*(a)\big) * tr_\theta\big(\chi_*(b)\big)\Big).$$

If we consider homology with coefficients in $\mathbb{Q}$, the quotient maps q_θ, q_ϑ induce surjective maps in homology. Thus there are uniquely determined classes $x, y \in H_*(\Lambda; \mathbb{Q})^\vartheta = H_*(\Lambda; \mathbb{Q})^\theta$ with $q_{\vartheta*}(x) = a$, $q_{\vartheta*}(y) = b$. It follows that

$$tr_\theta\big(\chi_*(a)\big) = tr_\theta\Big(\chi_*\big(q_{\vartheta*}(x)\big)\Big) = tr_\theta\Big(q_{\theta*}\big(\chi_{\frac{1}{4}*}(x)\big)\Big) = tr_\theta\big(q_{\theta*}(x)\big),$$

since $\chi_{\frac{1}{4}}$ is homotopic to the identity. Using the properties of the transfer homomorphisms we then get

$$tr_\theta\big(q_{\theta*}(x)\big) = x + \theta_*(x) = 2x = x + \vartheta_*(x) = tr_\vartheta\big(q_{\vartheta*}(x)\big) = tr_\vartheta(a),$$

as $\vartheta_* = \theta_*$. Therefore

$$\begin{aligned} P_\theta\big(\chi_*(a), \chi_*(b)\big) &= q_{\theta*}\Big(tr_\theta\big(\chi_*(a)\big) * tr_\theta\big(\chi_*(b)\big)\Big) \\ &= q_{\theta*}\big(tr_\vartheta(a) * tr_\vartheta(b)\big) \\ &= q_{\theta*} \circ \chi_{\frac{1}{4}*}\big(tr_\vartheta(a) * tr_\vartheta(b)\big) \\ &= (\chi_* \circ q_{\vartheta*})\big(tr_\vartheta(a) * tr_\vartheta(b)\big) \\ &= \chi_*\big(P_\vartheta(a, b)\big) \end{aligned}$$

and χ_* is an algebra homomorphism. It is an isomorphism since it inverse is also an algebra homomorphism as can be seen in the same way. □

COROLLARY 2.8. *The homomorphism*

$$(ev_0/\vartheta)_* : \big(H_*(\Lambda/\vartheta; \mathbb{Q}), P_\vartheta\big) \to \big(H_*(M; \mathbb{Q}), \bullet\big)$$

and

$$(ev_0/\theta)_* \circ (\chi_*)^{-1} : \big(H_*(\Lambda/\theta; \mathbb{Q}), P_\theta\big) \to \big(H_*(M; \mathbb{Q}), \bullet\big)$$

are algebra homomorphisms.

We close this section with a few words on how the transfer for a ramified covering is defined in [**Smi83**]:
Let X be any topological space and G and finite discrete topological group acting continuously on X. Then the orbit map $p : X \to X/G$ is a ramified covering in the sense of [**Smi83**]. In the case of a finite group action this means that p comes equipped with a so-called multiplicity function $\mu : X \to \mathbb{N}$ such that

(1) for all $a \in X/G$ we have

$$\sum_{x \in p^{-1}(a)} \mu(x) = |G|,$$

(2) if $p^{-1}(a) = \{x_1, \ldots, x_m\}$, $1 \leq m \leq |G|$, then the map

$$t_p : X/G \to SP^{|G|}(X),\ a \mapsto [x_1, \ldots, x_1, \ldots, x_m, \ldots, x_m],$$

where each x_i occurs $\mu(x_i)$ times in the unordered tupel $[x_1, \ldots, x_1, \ldots, x_m, \ldots, x_m]$, is continuous

(compare [**Smi83**]). Here $SP^d(A)$ denotes the d-fold symmetric product of a topological space A ([**AGP02**, Definition 5.2.1]).
We remark that the map t_p is explicitly given in Proposition 1.8 of [**Dol86**]. Proposition 1.9 of the same reference states that all ramified covering maps are given by the orbit map of a finite group action.

In Section 2 of [**Smi83**] a map $\widetilde{t_p} : X/G \to SP^\infty(X)$ into the infinite symmetric product $SP^\infty(X)$ of X is defined as the composition of t_p and an inclusion $SP^{|G|}(X) \hookrightarrow SP^\infty(X)$. If $SP^\infty(X)$ is path-connected then $SP^\infty(X)$ is weakly homotopy equivalent to a space K with the property that $\pi_n(K) = \pi_n\big(SP^\infty(X)\big)$ for $n \geq 1$ and $\pi_0(K) = \pi_0\big(SP^\infty(X)\big) = 0$. This follows form [**AGP02**, Theorem 6.4.15] or [**Hat02**, Corollary 4K.7]. (The reference [**Hat02**, Corollary 4K.7] actually demands $SP^\infty(X)$ to be an H-space, which might not hold for arbitrary spaces with the standard topology on $SP^\infty(X)$ (compare [**AGP02**, Appendix A.2.]). But if X is a countable CW-complex, then $SP^\infty(X)$ is an H-space, as Theorem A.2.5 in [**AGP02**] says.) From this it follows by the Dold-Thom theorem that for X path-connected and homotopy equivalent to a CW-complex, the $SP^\infty(X)$ is weakly homotopy equivalent to a space K having the property that $\pi_n(K) \cong H_n(X;\mathbb{Z})$ for $n \geq 1$ and $\pi_0(K) \cong \widetilde{H}_0(X;\mathbb{Z}) = 0$. This holds since in this case we have $\pi_i\big(SP^\infty(X)\big) = H_i(X;\mathbb{Z})$ for all $i \geq 1$. See Corollary 4K.7 and comment right below in [**Hat02**].
K actually is a product of Eilenberg-MacLane space of type $K(\tilde{H}_n(X;\mathbb{Z}), n)$. A weak homotopy equivalence is then enough to ensure that $\widetilde{t_p}$ defines a cohomology class

$$\big[\widetilde{t_p}\big] \in \big[X/G, K\big]_0 \cong \tilde{H}^*\big(X/G; \tilde{H}_*(X;\mathbb{Z})\big),$$

where we consider $\widetilde{t_p} : X/G \to K$ as a map into K by postcomposing it with the weak homotopy equivalence $SP^\infty(X) \to K$ ([**AP06**, Section 9]). Here $[A, B]_0$ denotes the set of pointed homotopy classes of maps between spaces A and B. The image of that class under the homomorphism

$$\tilde{H}^*\big(X/G; \tilde{H}_*(X;\mathbb{Z})\big) \longrightarrow Hom_{\mathbb{Z}}\big(\tilde{H}_*(X/G;\mathbb{Z}), \tilde{H}_*(X;\mathbb{Z})\big).$$

is then the desired transfer.

The refined definitions of transfer maps for ramified coverings presented in [**Dol86**] and [**AP06**] are also applicable in our case since Λ is compactly generated Hausdorff and has the homotopy type of a (countable) CW-complex. The transfer maps of all the three references [**Smi83**], [**Dol86**] and [**AP06**] satisfy the above properties (1)-(3) and they even agree for certain coefficients. Moreover, for finite group actions on a space X homotopy equivalent to a

CW-complex Dold's definition of the transfer is via a chain map on singular chains ([**Dol86**, Definition 2.6]).

3. Orientation reversal and shift: Construction of product A_θ

As described in Section 1, on Λ we have the $\mathbb{Z}_2$-action θ induced by the involution $\theta : \Lambda \to \Lambda :\ \gamma \mapsto \frac{1}{2}.\bar{\gamma}$, where $\bar{\gamma}(t) := \gamma(1-t)$ and $\frac{1}{2}.\gamma(t) = \gamma(t+\frac{1}{2})$. So $\theta(\gamma)(t) = \frac{1}{2}.\bar{\gamma}(t) = \gamma(1-t+\frac{1}{2})$. On $\Lambda \times \Lambda$ we define a $\mathbb{Z}_2$-action via the involution $(\gamma, \delta) \mapsto (\bar{\gamma}, \bar{\delta})$. With this definition the concatenation at time $\frac{1}{2}$ is $\mathbb{Z}_2$-equivariant as shown in Section 3.0.2 below.

The $\mathbb{Z}_2$-actions induced by the inovolutions

$$\theta : \Lambda \to \Lambda,\ \theta(\gamma) = \frac{1}{2}.\bar{\gamma},$$

$$\vartheta \times \vartheta : \Lambda \times \Lambda \to \Lambda \times \Lambda,\ (\vartheta \times \vartheta)(\gamma, \delta) = (\bar{\gamma}, \bar{\delta}),$$

will also be referred to as θ and $\vartheta \times \vartheta$ respectively. We denote both quotient maps by q:

$$q : \Lambda \to \Lambda/\mathbb{Z}_2 =: \Lambda/\theta,$$

$$q : \Lambda \times \Lambda \to (\Lambda \times \Lambda)/\mathbb{Z}_2 =: (\Lambda \times \Lambda)/(\vartheta \times \vartheta) = \Lambda^2/(\vartheta \times \vartheta).$$

The evaluation $ev_0 \times ev_0 : \Lambda^2 \to M^2$ is equivariant with respect to the $\mathbb{Z}_2$-action $\vartheta \times \vartheta$ on Λ^2 and the trivial action on M^2: $(ev_0 \times ev_0)\big((\vartheta \times \vartheta)(\gamma, \delta)\big) = (ev_0 \times ev_0)(\bar{\gamma}, \bar{\delta}) = \big(\bar{\gamma}(0), \bar{\delta}(0)\big) = \big(\gamma(0), \delta(0)\big) = (ev_0 \times ev_0)(\gamma, \delta)$. Hence it descends to the quotient: The diagram

$$\begin{tikzcd}
\Lambda^2 \arrow[d, "q"'] \arrow[dr, "ev_0 \times ev_0"] & \\
\Lambda^2/\theta \arrow[r, "(ev_0 \times ev_0)/(\vartheta \times \vartheta)"'] & M^2
\end{tikzcd}$$

commutes.
Note that $\mathcal{F}$ is invariant under $\vartheta \times \vartheta$, hence $i_{\mathcal{F}}/(\vartheta \times \vartheta) : \mathcal{F}/(\vartheta \times \vartheta) \hookrightarrow \Lambda^2/(\vartheta \times \vartheta)$ is a closed embedding, since the quotient maps q are closed maps: As $\mathbb{Z}_2$ is compact Hausdorff and $\mathcal{F}$ and Λ^2 are Hausdorff [**Bre72**, Chapter 1, Theorem 3.1.] applies and thus the quotients are Hausdorff and the quotient maps are closed maps. Given a closed subset $U \subset \mathcal{F}$, $q_{|\mathcal{F}}^{-1}(U)$ is closed by definition of the quotient topology. Thus $(q \circ i_{\mathcal{F}})\big(q_{|\mathcal{F}}^{-1}(U)\big) = (i_{\mathcal{F}}/(\vartheta \times \vartheta) \circ q_{|\mathcal{F}})\big(q_{|\mathcal{F}}^{-1}(U)\big) = (i_{\mathcal{F}}/(\vartheta \times \vartheta))(U)$ is closed since q and $i_{\mathcal{F}}$ are closed maps. As a continuous injective closed map $i_{\mathcal{F}}/(\vartheta \times \vartheta)$ is an embedding ([**Lee11**, Propositon 3.16.]).

Lemma 3.1. *Let $\Lambda = \Lambda M$ be the free loop space of a compact manifold M. The commutative diagram*

(3.1)
$$\begin{tikzcd}
\mathcal{F}/(\vartheta \times \vartheta) \arrow[r, hook, "i_{\mathcal{F}}/(\vartheta \times \vartheta) = i_{\mathcal{F}/(\vartheta \times \vartheta)}"] \arrow[d, "ev_0/(\vartheta \times \vartheta)"'] & (\Lambda \times \Lambda)/(\vartheta \times \vartheta) \arrow[d, "ev_0 \times ev_0/(\vartheta \times \vartheta)"] \\
M \arrow[r, hook, "\Delta"'] & M \times M
\end{tikzcd}$$

is a pullback diagram.

PROOF. Consider the pullback

$$\begin{tikzcd}
X_{\vartheta\times\vartheta} \arrow[r, hook, "pr_{\Lambda^2/(\vartheta\times\vartheta)}"] \arrow[d] & (\Lambda\times\Lambda)/(\vartheta\times\vartheta) \arrow[d, "ev_0\times ev_0/(\vartheta\times\vartheta)"] \\
M \arrow[r, hook, "\Delta"'] & M\times M
\end{tikzcd}$$

where $X_{\vartheta\times\vartheta} := (ev_0\times ev_0/(\vartheta\times\vartheta))^*(M) = \big\{\big(m,[(\gamma,\delta)]\big)\in M\times(\Lambda\times\Lambda)/(\vartheta\times\vartheta) \,|\, (m,m) = \big(\gamma(0),\delta(0)\big)\big\}$. The map $pr_{\Lambda^2/(\vartheta\times\vartheta)}$ is a closed embedding and we have $X_{\vartheta\times\vartheta}\cong\mathcal{F}/(\vartheta\times\vartheta)$. To see this, note that since q is surjective we have

$$\begin{aligned}
pr_{\Lambda^2/(\vartheta\times\vartheta)}(X_{\vartheta\times\vartheta}) &= (ev_0\times ev_0/(\vartheta\times\vartheta))^{-1}\big(\Delta(M)\big) = q\bigg(q^{-1}\Big((ev_0\times ev_0/(\vartheta\times\vartheta))^{-1}\big(\Delta(M)\big)\Big)\bigg)\\
&= q\big((ev_0\times ev_0/(\vartheta\times\vartheta)\circ q)^{-1}\big(\Delta(M)\big)\big) = q\big((ev_0\times ev_0)^{-1}\big(\Delta(M)\big)\big) = q(\mathcal{F})\\
&= \mathcal{F}/(\vartheta\times\vartheta).
\end{aligned}$$

The map $f_{\vartheta\times\vartheta}:\mathcal{F}/(\vartheta\times\vartheta)\to X_{\vartheta\times\vartheta},\ [(\gamma,\delta)]\mapsto\big(\gamma(0),[(\gamma,\delta)]\big)$ is then a homeomorphism, it fits into the following commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f", "\cong"'] \arrow[d] & M\times_{M^2}\Lambda^2 \arrow[r, hook, "pr_{\Lambda^2}"] \arrow[d, "g"'] & \Lambda^2 \arrow[d, "q"'] \arrow[dd, bend left=60, "ev_0\times ev_0"] \\
\mathcal{F}/(\vartheta\times\vartheta) \arrow[r, "f_{\vartheta\times\vartheta}", "\cong"'] \arrow[dr, "ev_0/(\vartheta\times\vartheta)"'] & X_{\vartheta\times\vartheta} \arrow[r, hook, "pr_{\Lambda^2/(\vartheta\times\vartheta)}"] \arrow[d] & \Lambda^2/(\vartheta\times\vartheta) \arrow[d, "ev_0\times ev_0/(\vartheta\times\vartheta)"'] \\
& M \arrow[r, hook, "\Delta"'] & M^2
\end{tikzcd}$$

where

$$\begin{aligned}
M\times_{M^2}\Lambda^2 = \Delta^*(\Lambda^2) &= (ev_0\times ev_0)^*(M) = (ev_0\times ev_0/(\vartheta\times\vartheta)\circ q)^*(M)\\
&\cong q^*\big((ev_0\times ev_0/(\vartheta\times\vartheta))^*(M)\big) = q^*(X_{\vartheta\times\vartheta}).
\end{aligned}$$

The map g is the composition

$$\big(m,(\gamma,\delta)\big)\mapsto\big(m,[(\gamma,\delta)],(\gamma,\delta)\big)\mapsto\big(m,[(\gamma,\delta)]\big)$$

where the first map is the homeomorphism between $M\times_{M^2}\Lambda^2$ and $q^*(X_{\vartheta\times\vartheta})$ and the second map in the projection onto $X_{\vartheta\times\vartheta}$. Hence $g = id_M\times q$ and, since M is (locally) compact, g is also a quotient map ([**LS15**, Folgerung 4.24]). f is then equivariant with equivariant inverse. Note that $f_{\vartheta\times\vartheta}\circ pr_{\Lambda^2/(\vartheta\times\vartheta)} = i_{\mathcal{F}/(\vartheta\times\vartheta)} = i_{\mathcal{F}}/(\vartheta\times\vartheta)$ is the inclusion of the subspace $\mathcal{F}/(\vartheta\times\vartheta)\subset\Lambda^2/(\vartheta\times\vartheta)$. $\square$

We wish to define a product

$$A_\theta : H_i(\Lambda/\theta;R)\times H_j(\Lambda/\theta;R)\to H_{i+j-n}(\Lambda/\theta;R)$$

for arbitrary coefficients R on $\Lambda = \Lambda M$ where M is a compact, connected, oriented n-manifold, by

$$\begin{tikzcd}
H_i(\Lambda/\theta; R) \times H_j(\Lambda/\theta; R) \arrow[d, "\text{transfer}\times\text{transfer}"'] \\
H_i(\Lambda; R) \times H_j(\Lambda; R) \arrow[d, "\times"'] \\
H_{i+j}(\Lambda \times \Lambda; R) \arrow[d] \\
H_{i+j}\big((\Lambda \times \Lambda)/(\vartheta \times \vartheta); R\big) \arrow[d] \\
H_{i+j}\big((\Lambda \times \Lambda)/(\vartheta \times \vartheta), (\Lambda \times \Lambda)/(\vartheta \times \vartheta) - \mathcal{F}/(\vartheta \times \vartheta); R\big) \arrow[d, "\text{Thom isomorphism (see 3.0.1)}"'] \\
H_{i+j-n}\big(\mathcal{F}/(\vartheta \times \vartheta); R\big) \arrow[d, "\text{concatenation (see 3.0.2)}"'] \\
H_{i+j-n}(\Lambda/\theta; R)
\end{tikzcd}$$

where we use the transfer map of the ramified covering $\Lambda \to \Lambda/\theta$ as explained in Section 2. We still have to define the two maps at the bottom:

3.0.1. *the Thom isomorphism.* As above, we pullback the normal bundle $N_M \to M$ along the map $ev_0/(\vartheta \times \vartheta)$ in the diagram 3.1. Let us, from now on, use just N to denote the normal bundle N_M of $M \cong \Delta(M)$ in M^2. We get a bundle over $\mathcal{F}/(\vartheta \times \vartheta)$:

$$\begin{tikzcd}
\big(ev_0/(\vartheta \times \vartheta)\big)^*(N) \arrow[r] \arrow[d] & N \arrow[d] \\
\mathcal{F}/(\vartheta \times \vartheta) \arrow[r, "ev_0/(\vartheta\times\vartheta)"'] & M
\end{tikzcd}$$

which fits into the larger diagram

$$\begin{tikzcd}
{ev_0}^*(N) \arrow[rr, "q\times id_N"'] \arrow[rrr, bend left, "pr_N"] \arrow[dr, "\cong"'] \arrow[dd] & & \big(ev_0/(\vartheta\times\vartheta)\big)^*(N) \arrow[r, "pr_N/((\vartheta\times\vartheta)\times id_N)"'] \arrow[dd] & N \arrow[dd] \\
& q^*\Big(\big(ev_0/(\vartheta\times\vartheta)\big)^*(N)\Big) \arrow[ur] \arrow[dl] & & \\
\mathcal{F} \arrow[rr, "q"] \arrow[rrr, bend right, "ev_0"'] & & \mathcal{F}/(\vartheta\times\vartheta) \arrow[r, "ev_0/(\vartheta\times\vartheta)"] & M
\end{tikzcd}$$

On ${ev_0}^*(N) = \mathcal{F}\times_M N = \big\{\big((\gamma,\delta),(x,v)\big)|\ \gamma(0)=x\big\}\subset \mathcal{F}\times N$ the involution is $(\vartheta\times\vartheta)\times id_N$. Hence no identifications in the fibres are made and the quotient remains a vector bundle. Note that N is locally compact since it is the tangent bundle of a finite dimensional manifold and hence a finite dimensional manifold itself. Thus the map $q\times id_N$ is also a quotient map. It remains a quotient map when restricted to ${ev_0}^*(N)$. We set $\widetilde{pr_N} := pr_N/((\vartheta\times\vartheta)\times id_N)$ since it is also just the restriction of the projection of $\mathcal{F}/(\vartheta\times\vartheta)\times N$ onto N.

We wish to have a tubular neighbourhood map $t_{\mathcal{F}/(\vartheta\times\vartheta)} : ev_0/(\vartheta\times\vartheta)^*(N)\to \Lambda^2/(\vartheta\times\vartheta)$ for which the diagram

$$\begin{tikzcd}
\big(ev_0/(\vartheta\times\vartheta)\big)^*(N) \arrow[r, hook, "t_{\mathcal{F}/(\vartheta\times\vartheta)}"] \arrow[d, "\widetilde{pr_N}"'] & (\Lambda\times\Lambda)/(\vartheta\times\vartheta) \arrow[d, "ev_0\times ev_0/(\vartheta\times\vartheta)"] \\
N \arrow[r, hook, "t_M"] & M\times M
\end{tikzcd}$$

commutes. Here we cannot claim the existence of tubular neighbourhood of $\mathcal{F}/(\vartheta\times\vartheta)$ in $\Lambda^2/(\vartheta\times\vartheta)$ since we do not know whether $\mathcal{F}/(\vartheta\times\vartheta)$ is an embedded submanifold, in fact neither $\mathcal{F}/(\vartheta\times\vartheta)$ nor $\Lambda^2/(\vartheta\times\vartheta)$ must be manifolds. Nevertheless, we now show that

$$t_{\mathcal{F}/(\vartheta\times\vartheta)} = t_{\mathcal{F}}/(\vartheta\times\vartheta)$$

works:

Proposition 3.2. *Let (M,g) be a compact Riemannian manifold and let $\Lambda=\Lambda M$ denote its free loop space. Then, the tubular neighbourhood map $t_{\mathcal{F}} : {ev_0}^*(N)\to\Lambda\times\Lambda$ of Section 2 of Chapter 2 is equivariant with respect to the actions $(\vartheta\times\vartheta)\times id_N$ on ${ev_0}^*(N)$ and $\vartheta\times\vartheta$ on Λ^2. It descends to a tubular neighbourhood map $t_{\mathcal{F}}/(\vartheta\times\vartheta)$ of $\mathcal{F}/(\vartheta\times\vartheta)\subset\Lambda^2/(\vartheta\times\vartheta)$ with image $U_{\mathcal{F}/(\vartheta\times\vartheta),\varepsilon} := \{[(\gamma,\delta)]\in\Lambda^2/(\vartheta\times\vartheta)|\ d_g\big(\gamma(0),\delta(0)\big)<\varepsilon\}$ for some $\varepsilon>0$. Moreover, the diagram*

$$\begin{tikzcd}
\big(ev_0/(\vartheta\times\vartheta)\big)^*(N) \arrow[r, hook, "t_{\mathcal{F}}/(\vartheta\times\vartheta)"] \arrow[d, "\widetilde{pr_N}"'] & \Lambda^2/(\vartheta\times\vartheta) \arrow[d, "(ev_0\times ev_0)/(\vartheta\times\vartheta)"] \\
N \arrow[r, hook, "t_M"] & M^2
\end{tikzcd} \tag{3.2}$$

is commutative and $\big((ev_0 \times ev_0)/(\vartheta\times\vartheta)\big)(U_{\mathcal{F}/(\vartheta\times\vartheta),\varepsilon}) = U_{M,\varepsilon}$.

Proof. The embedding $t'_{\mathcal{F}}$, given by $t'_{\mathcal{F}}(\gamma,\delta,v) := \big(\gamma, \lambda(\delta,v)\big)$ (see Section 2 of Chapter 2) is equivariant with respect to the actions $(\vartheta\times\vartheta)\times id_N$ and $\vartheta\times\vartheta$:

$$\overline{\lambda(\delta,v)}(t) = h\big(\delta(0), \exp(\delta(0),v)\big)(\delta(1-t)) = h\big(\delta(0), \exp(\delta(0),v)\big)(\bar{\delta}(t)) = \lambda(\bar{\delta},v)(t),$$

so that

$$\begin{tikzcd}
(\gamma,\delta,v) \arrow[r, mapsto, "t'_{\mathcal{F}}"] \arrow[d, mapsto, "(\vartheta\times\vartheta)\times id_N"'] & \big(\gamma, \lambda(\delta,v)\big) \arrow[d, mapsto, "(\vartheta\times\vartheta)"] \\
(\bar{\gamma},\bar{\delta},v) \arrow[r, mapsto, "t'_{\mathcal{F}}"] & \big(\bar{\gamma}, \lambda(\bar{\delta},v)\big) = \big(\bar{\gamma}, \overline{\lambda(\delta,v)}\big)
\end{tikzcd}$$

commutes. Hence, if we identify $(\gamma,\lambda)\sim(\bar{\gamma},\bar{\lambda})$ we get a well-defined continuous function

$$\begin{aligned} t'_{\mathcal{F}}/(\vartheta\times\vartheta) :& ev_0{}^*(D_\varepsilon N) \to (\Lambda\times\Lambda)/(\vartheta\times\vartheta) \\ & \big([(\gamma,\delta)], v\big) \mapsto \big[\big(\gamma, \lambda(\delta,v)\big)\big], \end{aligned}$$

making the diagram

$$\begin{tikzcd}
ev_0{}^*(D_\varepsilon N) \arrow[r, hook, "t'_{\mathcal{F}}"] \arrow[d, "q\times id_N"'] & \Lambda\times\Lambda \arrow[d, "q"] \\
\big(ev_0/(\vartheta\times\vartheta)\big)^*(D_\varepsilon N) \arrow[r, hook, "t'_{\mathcal{F}}/(\vartheta\times\vartheta)"] \arrow[d, "\widetilde{pr_N}"'] & (\Lambda\times\Lambda)/(\vartheta\times\vartheta) \arrow[d, "ev_0\times ev_0/(\vartheta\times\vartheta)"] \\
D_\varepsilon N \arrow[r, hook, "t'_M"] & M\times M
\end{tikzcd} \tag{3.3}$$

commute.
We check that $t'_{\mathcal{F}}/(\vartheta\times\vartheta)$ has the desired properties:

- $t'_{\mathcal{F}}/(\vartheta\times\vartheta)$ is a homeomorphism onto its image

  $$U_{\mathcal{F}/(\vartheta\times\vartheta),\varepsilon} = \{[(\gamma,\delta)]\in\Lambda^2/(\vartheta\times\vartheta) \mid d_g\big(\gamma(0),\delta(0)\big) < \varepsilon\}.$$

  As q is an open map (being an orbit space projection) $U_{\mathcal{F}/(\vartheta\times\vartheta),\varepsilon} = q(U_{\mathcal{F},\varepsilon})$ is open and contains $\mathcal{F}/(\vartheta\times\vartheta)$. Since $\bar{\delta}(t) = \delta(1-t) = h\big(\gamma(0),\lambda(0)\big)^{-1}\big(\lambda(1-t)\big) = h\big(\gamma(0),\lambda(0)\big)^{-1}\big(\bar{\lambda}(t)\big)$, we have that

  $$\begin{tikzcd}
(\gamma,\delta,v) \arrow[d, mapsto, "(\vartheta\times\vartheta)\times id_N"'] & (\gamma,\lambda) \arrow[l, mapsto, "\kappa"'] \arrow[d, mapsto, "(\vartheta\times\vartheta)"] \\
(\bar{\gamma},\bar{\delta},v) & (\bar{\gamma},\bar{\lambda}) \arrow[l, mapsto, "\kappa"']
\end{tikzcd}$$

  commutes and so $\kappa/(\vartheta\times\vartheta) : U_{\mathcal{F}/(\vartheta\times\vartheta),\varepsilon} \to ev_0/(\vartheta\times\vartheta)^*(D_\varepsilon N)$

  $$\kappa/(\vartheta\times\vartheta)\big([(\gamma,\lambda)]\big) := \big([\gamma,\delta], v\big),$$

  with δ and v as defined in Section 2 of Chapter 2, is a continuous inverse of $t'_{\mathcal{F}}/(\vartheta\times\vartheta)$.
- Restricted to the zero section $t'_{\mathcal{F}}/(\vartheta\times\vartheta)$ coincides with the inclusion $i_{\mathcal{F}/(\vartheta\times\vartheta)} = i_{\mathcal{F}}/(\vartheta\times\vartheta)$ of $\mathcal{F}/(\vartheta\times\vartheta)$ into $\Lambda^2/(\vartheta\times\vartheta)$. This holds since $h(p,p) = id_M$.

- Precomposing with the fibrewise homeomorphism

$$\big(ev_0/(\vartheta \times \vartheta)\big)^*(N) \cong \big(ev_0/(\vartheta \times \vartheta)\big)^*(D_\varepsilon N)$$

  then yields the tubular neighbourhood map $t_{\mathcal{F}}/(\vartheta \times \vartheta)$.

Let us mention at this point that another piece of basic topology can be used to directly show that $t'_{\mathcal{F}}/(\vartheta \times \vartheta)$ is an open embedding: We have noted that $q \times id_N : {ev_0}^*(D_\varepsilon N) \to ev_0/(\vartheta \times \vartheta)^*(D_\varepsilon N)$ is a quotient map making the same identifications as the orbit map of the diagonal action, hence their quotients are uniquely homeomorphic. It follows that $q \times id_N$ is an open map. Thus in the diagram (3.3) the three maps at the top are open maps. As above $U \subset ev_0/(\vartheta \times \vartheta)^*(D_\varepsilon N)$ is open if and only if $(q \times id_N)^{-1}(U)$ is open. Hence $\big(t'_{\mathcal{F}}/(\vartheta \times \vartheta)\big)(U) = (t'_{\mathcal{F}}/(\vartheta \times \vartheta) \circ q \times id_N\big)\big((q \times id_N)^{-1}(U)\big) = q \circ t'_{\mathcal{F}}\big((q \times id_N)^{-1}(U)\big)$ is open and $t'_{\mathcal{F}}/(\vartheta \times \vartheta)$ is an open map. Thus $t'_{\mathcal{F}}/(\vartheta \times \vartheta)$ is an embedding since a continuous, injective open map is an embedding. □

We define $\tau_{\mathcal{F}/(\vartheta \times \vartheta)} := \widetilde{pr_N}^*(\tau_M)$, where τ_M is the Thom class of the normal bundle N of $\Delta(M) \subset M^2$. $\tau_{\mathcal{F}/(\vartheta \times \vartheta)}$ is a Thom class of the vector bundle $\big(ev_0/(\vartheta \times \vartheta)\big)^*(N) \to \mathcal{F}/(\vartheta \times \vartheta)$ ([**AGP02**, Proposition 11.7.11]). It follows from the above proof that $\tau_{\mathcal{F}} = pr_N^*(\tau_M) = (q \times id_N)^*\big(\widetilde{pr_N}^*(\tau_M)\big) = (q \times id_N)^*(\tau_{\mathcal{F}/(\vartheta \times \vartheta)})$.
Let $tr : H_i\big(\Lambda/(\vartheta \times \vartheta); R\big) \to H_i(\Lambda; R)$ be the transfer map. We can now define the product announced at the beginning of this section:

Definition 3.3. Let M be a compact, connected, oriented smooth n-manifold and let $\Lambda = \Lambda M$ denote its free loop space. On the homology $H_*(\Lambda/\theta; R)$ with arbitrary coefficients R we define the product

$$A_\theta : H_i(\Lambda/\theta; R) \times H_j(\Lambda/\theta; R) \to H_{i+j-n}(\Lambda/\theta; R) \tag{3.4}$$

to be the the composition down the middle of the following commutative diagram (coefficients omitted from notation).

(3.5)

$$\begin{tikzcd}
 & A_\theta \arrow[d] & \\
 & H_i(\Lambda/\theta)\times H_j(\Lambda/\theta) \arrow[dl, "tr\times tr"'] \arrow[dd, "q_*\circ\times\circ tr\times tr"] & \\
H_i(\Lambda)\times H_j(\Lambda) \arrow[d, "\times"'] & & H_i(M)\times H_j(M) \arrow[d, "\times"] \\
H_{i+j}(\Lambda^2) \arrow[r, "q_*"] \arrow[d] & H_{i+j}(\Lambda^2/(\vartheta\times\vartheta)) \arrow[r, "\big((ev_0\times ev_0)/(\vartheta\times\vartheta)\big)_*"] \arrow[d] & H_{i+j}(M^2) \arrow[d] \\
H_{i+j}(\Lambda^2,\Lambda^2-\mathcal{F}) \arrow[r, "q_*"] \arrow[d, "\cong"'] & H_{i+j}\big(\frac{\Lambda^2}{(\vartheta\times\vartheta)},\frac{\Lambda^2}{(\vartheta\times\vartheta)}-\frac{\mathcal{F}}{(\vartheta\times\vartheta)}\big) \arrow[r, "\big((ev_0\times ev_0)/(\vartheta\times\vartheta)\big)_*"] \arrow[d, "\cong"'] & H_{i+j}\big(M^2,M^2-\Delta(M)\big) \arrow[d, "\cong"'] \\
H_{i+j}(U_{\mathcal{F},\varepsilon},U_{\mathcal{F},\varepsilon}-\mathcal{F}) \arrow[r, "q_*"] \arrow[d, "\cong"', "(t_{\mathcal{F}*})^{-1}"] & H_{i+j}\big(U_{\frac{\mathcal{F}}{(\vartheta\times\vartheta)},\varepsilon},U_{\frac{\mathcal{F}}{(\vartheta\times\vartheta)},\varepsilon}-\frac{\mathcal{F}}{(\vartheta\times\vartheta)}\big) \arrow[r, "\big((ev_0\times ev_0)/(\vartheta\times\vartheta)\big)_*"] \arrow[d, "\cong"', "(t_{\mathcal{F}}/(\vartheta\times\vartheta)_*)^{-1}"] & H_{i+j}\big(U_M,U_M-\Delta(M)\big) \arrow[d, "\cong"', "(t_{M*})^{-1}"] \\
H_{i+j}(N_{\mathcal{F}},N_{\mathcal{F}}-\mathcal{F}) \arrow[r, "(q\times id_N\circ g)_*"'] \arrow[d, "\cong"', "\cap\tau_{\mathcal{F}}"] & H_{i+j}\Big(\big(\frac{ev_0}{(\vartheta\times\vartheta)}\big)^*(N),\big(\frac{ev_0}{(\vartheta\times\vartheta)}\big)^*(N)-\frac{\mathcal{F}}{(\vartheta\times\vartheta)}\Big) \arrow[r, "\widetilde{pr}_{N*}"'] \arrow[d, "\cong"', "\cap\tau_{\mathcal{F}/(\vartheta\times\vartheta)}"] & H_{i+j}(N,N-M) \arrow[d, "\cong"', "\cap\tau_M"] \\
H_{i+j-n}(N_{\mathcal{F}}) \arrow[r, "(q\times id_N\circ g)_*"'] \arrow[d, "\cong"'] & H_{i+j-n}\big(ev_0/(\vartheta\times\vartheta)^*(N)\big) \arrow[r, "\widetilde{pr}_{N*}"'] \arrow[d, "\cong"'] & H_{i+j-n}(N) \arrow[d, "\cong"'] \\
H_{i+j-n}(\mathcal{F}) \arrow[r, "q_*"'] \arrow[d, "\phi_*"'] & H_{i+j-n}(\mathcal{F}/(\vartheta\times\vartheta)) \arrow[r, "ev_0/(\vartheta\times\vartheta)_*"'] \arrow[d, "(\phi/\theta)_*"'] & H_{i+j-n}(M) \\
H_{i+j-n}(\Lambda) \arrow[r, "q_*"'] & H_{i+j-n}(\Lambda/\theta) &
\end{tikzcd}$$

where the map ϕ/θ is defined just below.

We make two observations:

- For $a, b \in H_*(\Lambda/\theta; R)$, we have

$$A_\theta(a,b) = q_*\big(tr(a) * tr(b)\big) = P_\theta(a,b),$$

  so A_θ is nothing but the transfer product defined in Section 2. This also shows that A_θ is independet of the metric g on M chosen to define $t_{\mathcal{F}}/(\vartheta\times\vartheta)$.
  The transfer product P_θ associated to the $\mathbb{Z}_2$-action induced by the involution θ can thus be realized as a "Chas-Sullivan-like" product using the concatenation of loops on the quotient $\mathcal{F}/(\vartheta\times\vartheta)$.

- The product is no longer in relation to the intersection product on the base: The evaluation $ev_0 : \Lambda \to M$ is not equivariant with respect to θ and the trivial action on M.

We summarize this as

THEOREM 3.4. *Let M be a compact, connected, oriented manifold and let $\theta : \Lambda M \to \Lambda M$ be the involution $\gamma \mapsto \frac{1}{2}.\bar{\gamma}$ that reverses the orientation of loops and shifts their starting points by $\frac{1}{2}$, then on $H_*(\Lambda/\theta; R)$ we have*

$$A_\theta = P_\theta \tag{3.6}$$

for arbitrary coefficients R.

REMARK 3.5. The vector bundle $ev_0/(\vartheta \times \vartheta)^*(N) \to \mathcal{F}/(\vartheta \times \vartheta)$ is orientable since the involution $(\vartheta \times \vartheta) \times id_N$ is a bundle isomorphism of the $\mathbb{Z}_2$-vector bundle $p : {ev_0}^*(N) \to \mathcal{F}$ with the property

$$((\vartheta \times \vartheta) \times id_N)^*(\tau_{\mathcal{F}}) = \tau_{\mathcal{F}}. \tag{3.7}$$

To see this consider the following commutative square

$$\begin{tikzcd}
H^n({ev_0}^*(N), {ev_0}^*(N) - \mathcal{F}) \arrow[r, "\big((\vartheta\times\vartheta)\times id_N\big)^*"] \arrow[d, "{j_{(\vartheta\times\vartheta)(\gamma,\delta)}}^*"] & H^n({ev_0}^*(N), {ev_0}^*(N) - \mathcal{F}) \arrow[d, "{j_{(\gamma,\delta)}}^*"] \\
H^n\big(p^{-1}\big((\vartheta\times\vartheta)(\gamma,\delta)\big), p^{-1}\big((\vartheta\times\vartheta)(\gamma,\delta)\big) - 0\big) \arrow[r, "\Big(\big((\vartheta\times\vartheta)\times id_N\big)_{(\gamma,\delta)}\Big)^*"] & H^n\big(p^{-1}\big(\gamma,\delta\big)\big), p^{-1}\big((\gamma,\delta)\big) - 0\big)
\end{tikzcd}$$

where $j_{(\gamma,\delta)} : p^{-1}\big((\gamma,\delta)\big) = \{(\gamma,\delta)\} \times N_{\gamma(0)} \hookrightarrow {ev_0}^*(N)$ denotes the inclusion of the fibre over (γ,δ). Since $(\vartheta \times \vartheta) \times id_N$ is fibrewise the identity, the lower horizontal map $\big(((\vartheta \times \vartheta) \times id_N)_{(\gamma,\delta)}\big)^*$ in the above diagram maps prescribed generator to prescribed generator and hence the upper horizontal map sends the Thom class $\tau_{\mathcal{F}}$ to itself. More precisely, $\tau_{\mathcal{F}}$ corresponds to the pullback Thom class $(pr_N)^*(\tau_M)$ where τ_M again denotes the Thom class of $N \to M$. As

$$\begin{tikzcd}
p^{-1}\big((\vartheta\times\vartheta)(\gamma,\delta)\big) \arrow[dr, "pr_{N(\bar{\gamma},\bar{\delta})}"'] & & p^{-1}\big((\gamma,\delta)\big) \arrow[ll, "\big((\vartheta\times\vartheta)\times id_N\big)_{(\gamma,\delta)}"'] \arrow[dl, "pr_{N(\gamma,\delta)}"] \\
& N_{\gamma(0)} &
\end{tikzcd}$$

commutes, we have

$$\begin{aligned}
(pr_N)^*(\tau_M)_{|(\gamma,\delta)} &= (pr_{N(\gamma,\delta)})^*(\tau_{M|\gamma(0)}) = \big(((\vartheta \times \vartheta) \times id_N)_{(\gamma,\delta)}\big)^*\big((pr_{N(\bar{\gamma},\bar{\delta})})^*(\tau_{M|\gamma(0)})\big) \\
&= (pr_N)^*(\tau_M)_{|(\vartheta\times\vartheta)(\gamma,\delta)}.
\end{aligned}$$

As ${j_{(\gamma,\delta)}}^*$ is an isomorphism in degree n =dimension of the fibre (it maps generator to generator), the conclusion follows.

3.0.2. *concatenation.* Concatenation is a continuous map $\phi : \mathcal{F} \to \Lambda$ defined by

$$\phi(\gamma,\delta)(t) = \phi_{\frac{1}{2}}(\gamma,\delta)(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \delta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.$$

We like to write $\gamma \cdot \delta$ for $\phi_{\frac{1}{2}}(\gamma,\delta)$.

The action θ on Λ maps $\gamma \cdot \delta$ to $\frac{1}{2}.\overline{\gamma \cdot \delta}$ which equals $\bar{\gamma} \cdot \bar{\delta}$:

$$\overline{\gamma \cdot \delta}(t) = \tag{3.8}$$

$$\overline{\phi(\gamma,\delta)}(t) = \phi(\gamma,\delta)(1-t) = \begin{cases} \gamma(2(1-t)), & 0 \le 1-t \le \frac{1}{2} \\ \delta(2(1-t)-1), & \frac{1}{2} \le 1-t \le 1 \end{cases} = \begin{cases} \gamma(1-2t+1), & \frac{1}{2} \le t \le 1 \\ \delta(1-2t), & 0 \le t \le \frac{1}{2} \end{cases}$$
$$= \begin{cases} \bar{\delta}(2t), & 0 \le t \le \frac{1}{2} \\ \bar{\gamma}(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} = \phi(\bar{\delta},\bar{\gamma})(t)$$
$$= \bar{\delta} \cdot \bar{\gamma},$$

$$\frac{1}{2}.\gamma \cdot \delta(t) = \gamma \cdot \delta(t+\tfrac{1}{2}) = \begin{cases} \gamma(2t+1), & 0 \le t+\frac{1}{2} \le \frac{1}{2} \\ \delta(2t), & \frac{1}{2} \le t+\frac{1}{2} \le 1 \end{cases} = \delta \cdot \gamma.$$

Hence, the concatenation $\phi = \phi_{\frac{1}{2}}$ at time $\frac{1}{2}$ is equivariant:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \Lambda \\ {\scriptstyle \vartheta\times\vartheta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathcal{F} & \xrightarrow{\phi} & \Lambda \end{array} \qquad \begin{array}{ccc} (\gamma,\delta) & \longmapsto & \gamma\cdot\delta \\ \downarrow & & \downarrow \\ (\bar{\gamma},\bar{\delta}) & \longmapsto & \bar{\gamma}\cdot\bar{\delta} \end{array}$$

We therefore get a map $\phi/\theta : \mathcal{F}/(\vartheta \times \vartheta) \to \Lambda/\theta$, defined by $\phi/\theta \circ q = q \circ \phi$. The induced mapping on homology $\phi/\theta_* : H_i(\mathcal{F}/(\vartheta \times \vartheta)) \to H_i(\Lambda/\theta)$ thus satisfies

$$\phi/\theta_* \circ q_* = q_* \circ \phi_*$$

REMARK 3.6. We can also concatenate at a different time s:

$$\phi_s(\gamma,\delta)(t) = \begin{cases} \gamma(\frac{t}{s}) & 0 \le t \le s \\ \delta(\frac{t-s}{1-s}) & s \le t \le 1 \end{cases}.$$

These concatenations are all homotopic, i.e. $\phi \simeq \phi_s$ and hence induce the same maps in homology. Thus

$$\phi/\theta_* \circ q_* = q_* \circ \phi_* = q_* \circ \phi_{s*}$$

for any $s \in (0,1)$. However, on the left-hand side of this equation we cannot simply replace ϕ/θ with ϕ_s/θ. ϕ_s/θ is in fact not defined since ϕ_s it is only equivariant up to homotopy.

4. Orientation reversal: Construction of product A_ϑ

On Λ we have the $\mathbb{Z}_2$-action ϑ: $\gamma \mapsto \bar{\gamma}$, where $\bar{\gamma}(t) := \gamma(1-t)$.
On $\Lambda \times \Lambda$ we define the $\mathbb{Z}_2$-action ψ induced by the involution $(\gamma, \delta) \mapsto (\bar{\delta}, \bar{\gamma})$. With this definition the concatenation at time $\frac{1}{2}$ is equivariant, see Section 4.0.2 below.
The corresponding involutions have the same names:

$$\vartheta : \Lambda \to \Lambda,\ \vartheta(\gamma) = \bar{\gamma},$$

$$\psi : \Lambda \times \Lambda \to \Lambda \times \Lambda,\ \psi(\gamma, \delta) = (\bar{\delta}, \bar{\gamma}).$$

Both qoutient maps are denoted by q:

$$q : \Lambda \to \Lambda/\mathbb{Z}_2 =: \Lambda/\vartheta,$$

$$q : \Lambda \times \Lambda \to (\Lambda \times \Lambda)/\mathbb{Z}_2 =: (\Lambda \times \Lambda)/\psi = \Lambda^2/\psi.$$

Let $\mathrm{T} : \Lambda^2 \to \Lambda^2$ be the transposition map: $\mathrm{T}(\gamma, \delta) = (\delta, \gamma)$. For the induced map on $\mathcal{F}$ we also use the notation T. Obviously T is also an involution and following relations hold

$$\psi = \mathrm{T} \circ (\vartheta \times \vartheta) = (\vartheta \times \vartheta) \circ \mathrm{T}.$$

This should be compared to equation (1.1) of Section 1.

It seems that we have to introduce this twist T to get a $\mathbb{Z}_2$-action on Λ^2 such that the concatenation ϕ is equivariant with respect to the orientation reversal ϑ on Λ. It is this twist that makes the construction of a product more complicated than in the case of the action θ.

If we consider the $\mathbb{Z}_2$-action on $M \times M$ induced by the involution $(x, y) \mapsto (y, x)$, which we will also denote by T, then $ev_0 \times ev_0 : \Lambda^2 \to M^2$ is equivariant with respect to the action $\psi = \mathrm{T} \circ (\vartheta \times \vartheta)$ on Λ^2 and the action T on M.

Lemma 4.1. *Let $\Lambda = \Lambda M$ be the free loop space of a compact manifold M. The commutative diagram*

$$\begin{tikzcd} \mathcal{F}/\psi \arrow[r, hook, "i_{\mathcal{F}}/\psi = i_{\mathcal{F}/\psi}"] \arrow[d, "ev_0/\psi"'] & (\Lambda \times \Lambda)/\vartheta \arrow[d, "ev_0 \times ev_0/\psi"] \\ M \arrow[r, hook, "\Delta/\mathrm{T}"'] & (M \times M)/\mathrm{T} \end{tikzcd} \tag{4.1}$$

is a pullback diagram.

Proof. We have a pullback

$$\begin{tikzcd} X_\psi \arrow[r, hook, "pr_{\Lambda^2/\psi}"] \arrow[d] & (\Lambda \times \Lambda)/\psi \arrow[d, "ev_0 \times ev_0/\psi"] \\ M \arrow[r, hook, "\Delta/\mathrm{T}"'] & (M \times M)/\mathrm{T} \end{tikzcd}$$

where $X_\psi := \big\{\big(m, [(\gamma, \delta)]\big) \in M \times (\Lambda \times \Lambda)/\psi |\ (m, m) = \big(\gamma(0), \delta(0)\big)\big\} \cong \mathcal{F}/\psi$. That $\mathcal{F}/\psi \cong X_\psi \cong (M \times_{M^2} \Lambda^2)/\mathbb{Z}_2$ holds by the following reasoning:
Consider on $M \times \Lambda^2$ the involution $(id_M \times \psi)\big(m, (\gamma, \delta)\big) = (m, \bar{\delta}, \bar{\gamma})$ under which $M \times_{M^2} \Lambda^2$

is invariant. Considering also the involutions T on M^2 and the trivial involution on M the whole pullback

$$\begin{tikzcd}
M \times_{M^2} \Lambda^2 \arrow[r, hook] \arrow[d] & \Lambda^2 \arrow[d, "ev_0 \times ev_0"] \\
M \arrow[r, hook, "\Delta"'] & M^2
\end{tikzcd}$$

is equivariant and we get a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[dr, "f", "\cong"'] \arrow[rrrr, hook, bend left] \arrow[dddr, bend right, "ev_0"'] & & & & \\
& M \times_{M^2} \Lambda^2 \arrow[rrr, hook, "pr_{\Lambda^2}"] \arrow[dr, "q_{id_M \times \psi}"] \arrow[ddd, "pr_M"'] & & & \Lambda^2 \arrow[d, "q"] \\
& & (M \times_{M^2} \Lambda^2)/(id_M \times \psi) \arrow[rr, hook, "pr_{\Lambda^2}/(id_M \times \psi)"] \arrow[dr, "h", "\cong"'] \arrow[dd, "pr_M/(id_M \times \psi)"'] & & \Lambda^2/\psi \arrow[dd, "ev_0 \times ev_0/\psi"] \\
& & & X_\psi = M \times_{M^2/\mathrm{T}} \Lambda^2/\psi \arrow[ur, hook, "pr_{\Lambda^2/\psi}"'] \arrow[dl] & \\
& M \arrow[r, "="] & M \arrow[rr, hook, "\Delta/\mathrm{T}"'] & & M^2/\mathrm{T}
\end{tikzcd}$$

Here h is a homeomorphism for the same reason f is, namely h is bijective, continuous and open: It is given by

$$\begin{aligned}
h\Big(\big[\big(m,(\gamma,\delta)\big)\big]\Big) &:= \bigg(pr_M/(id_M \times \psi)\Big(\big[\big(m,(\gamma,\delta)\big)\big]\Big), pr_{\Lambda^2}/(id_M \times \psi)\Big(\big[\big(m,(\gamma,\delta)\big)\big]\Big)\bigg) \\
&= \big(m, [(\gamma,\delta)]\big)
\end{aligned}$$

which is clearly continuous.
Since $(h \circ q_{id_M \times \psi})\big(m,(\gamma,\delta)\big) = \big(m,[(\gamma,\delta)]\big)$, we see that $h \circ q_{id_M \times \psi} = id_M \times q$, which is an open map. Hence, if $U \subset (M \times_{M^2} \Lambda^2)/(id_M \times \psi)$ is open, we have $h(U) = (h \circ q_{id_M \times \psi})\big((q_{id_M \times \psi})^{-1}(U)\big) = (id_M \times q)\big((q_{id_M \times \psi})^{-1}(U)\big)$ which is open since $(q_{id_M \times \psi})^{-1}(U)$ is open (by definition of the quotient topology). Its image is obviously all of X_ψ since $\big(m,[(\gamma,\delta)]\big) \in X_\psi \Leftrightarrow m = \gamma(0) = \delta(0) \Leftrightarrow \big(m,(\gamma,\delta)\big) \in M \times_{M^2} \Lambda^2$ it follows that $(h \circ q_{id_M \times \psi})\big(m,(\gamma,\delta)\big) = \big(m,[(\gamma,\delta)]\big)$. As $q_{id_M \times \psi}$ identifies (m,γ,δ) with $(m,\bar{\delta},\bar{\gamma})$, h is also injective.
The homeomophism $f : \mathcal{F} \to M \times_{M^2} \Lambda^2$, $(\gamma,\delta) \mapsto \big(\gamma(0),(\gamma,\delta)\big)$ and its inverse $f^{-1}\big(m,(\gamma,\delta)\big) \mapsto (\gamma,\delta)$ are obviously equivariant:

$$\begin{tikzcd}
(\gamma,\delta) \arrow[r, mapsto, shift left, "f"] \arrow[d, mapsto, "\psi"'] & \big(\gamma(0),(\gamma,\delta)\big) \arrow[l, mapsto, shift left, "f^{-1}"] \arrow[d, mapsto, "id_M \times \psi"] \\
(\bar{\delta},\bar{\gamma}) \arrow[r, mapsto, shift left, "f"] & \big(\gamma(0),(\bar{\delta},\bar{\gamma})\big). \arrow[l, mapsto, shift left, "f^{-1}"]
\end{tikzcd}$$

Hence $\mathcal{F}/\psi \cong X_\psi$ via f/ψ. In other words, the commutative square

$$\begin{tikzcd}
\mathcal{F}/\psi \arrow[r, hook, "i_{\mathcal{F}/\psi}"] \arrow[d, "ev_0/\psi"'] & (\Lambda \times \Lambda)/\psi \arrow[d, "ev_0 \times ev_0/\psi"] \\
M \arrow[r, hook, "\Delta/\mathrm{T}"'] & (M \times M)/\mathrm{T}
\end{tikzcd}$$

is ("homeomorphic to") a pullback square. We again observe that $i_\mathcal{F}/\psi : \mathcal{F}/\psi \to \Lambda^2/\psi$ is a closed embedding and $i_\mathcal{F}/\psi = i_{\mathcal{F}/\psi} = pr_{\Lambda^2/\psi} \circ h \circ f/\psi$. □

We wish to define a product

$$A_\vartheta : H_i(\Lambda/\vartheta; R) \times H_j(\Lambda/\vartheta; R) \to H_{i+j-n}(\Lambda/\vartheta; R)$$

on singular homology with arbitrary coefficients R of $\Lambda = \Lambda M$, where M is a compact, connected, oriented n-manifold, by

$$\begin{tikzcd}
H_i(\Lambda/\vartheta; R) \times H_j(\Lambda/\vartheta; R) \arrow[d, "\text{transfer}\times\text{transfer}"'] \\
H_i(\Lambda; R) \times H_j(\Lambda; R) \arrow[d, "\times"'] \\
H_{i+j}(\Lambda \times \Lambda; R) \arrow[d] \\
H_{i+j}\big((\Lambda \times \Lambda)/\psi; R\big) \arrow[d] \\
H_{i+j}\big((\Lambda \times \Lambda; R)/\psi, (\Lambda \times \Lambda)/\psi - \mathcal{F}/\psi; R\big) \arrow[d, "\text{Thom isomorphism (see 4.0.1)}"'] \\
H_{i+j-n}(\mathcal{F}/\psi; R) \arrow[d, "\text{concatenation (see 4.0.2)}"'] \\
H_{i+j-n}(\Lambda/\vartheta; R)
\end{tikzcd}$$

where we use the transfer map of the ramified covering $\Lambda \to \Lambda/\vartheta$ as explained in Section 2. We still have to define the two maps at the bottom:

4.0.1. *the Thom isomorphism.* The diagonal $\Delta(M) \subset M \times M$ is fixed under the involution T, in fact $\mathrm{Fix}(\mathrm{T}) = \Delta(M)$. We thus have a commutative diagram

$$\begin{tikzcd}
M \arrow[r, hook, "\Delta"] \arrow[dr, hook, "\Delta/\mathrm{T}"'] & M \times M \arrow[d, "q"] \\
& (M \times M)/\mathrm{T}.
\end{tikzcd}$$

Δ is a closed map (since M is Hausdorff, $\Delta(M)$ is closed in $M\times M$) and q is a closed map (since $\mathbb{Z}_2$ is compact Hausdorff and M is Hausdorff, see e.g.[**LS15**, Satz 5.7]). Thus $\Delta/\mathrm{T} := q\circ\Delta$ is also closed. Since Δ/T is in addition injective and continuous, it is an embedding. We can thus ask whether its image $(\Delta/\mathrm{T})(M)\cong\Delta(M)$ has a tubular neighbourhood in $(M\times M)/\mathrm{T}$ that lifts to a tube around $\mathcal{F}/\vartheta$ inside $(\Lambda\times\Lambda)/\vartheta$.

Let $N\to M$ again be the normal bundle of $\Delta(M)\subset M^2$. Consider now the symmetrized tubular neighbourhood embedding $s'_M : D_\varepsilon N\hookrightarrow M\times M, (x,v)\mapsto \big(\exp(x,-v),\exp(x,v)\big)$ constructed in Section 2 of Chapter 2 and the fibre preserving involution $\mathrm{t}: N\to N,\ (x,v)\mapsto (x,-v)$. s'_M is equivariant with respect to the action induced by t on N and the action T on $M\times M$ and we thus get a commutative diagram

$$\begin{tikzcd} N \arrow[r, hook, "s_M"] \arrow[d, "q_{\mathrm{t}}"'] & M\times M \arrow[d, "q_{\mathrm{T}}"] \\ N/\mathrm{t} \arrow[r, hook, "s_M/\mathrm{T}"'] & (M\times M)/\mathrm{T} \end{tikzcd}$$

where N/t is the quotient space of N. Since s_M and the quotient maps q are open, so is the induced map s_M/T. Since it also injective and continuous, it is an embedding.

Lemma 4.2. *Let M be a simply-connected compact manifold. The map $p/\mathrm{t}: N/\mathrm{t}\to M$, induced by the bundle projection $p: N\to M$, is a fibre bundle with typical fibre $\mathbb{R}^n/(v\sim -v)$. s_M/T embeds N/t into M^2/T as an open neighbourhood of the diagonal $\Delta(M)$. On the zero section s_M/T restricts to the inclusion $M\hookrightarrow\Delta(M)\subset M^2/\mathrm{T}$.*
Moreover, if $n = rank(N) = dim(M)$ is even, then there exists a rational Thom class $\omega\in H^n(N/\mathrm{t}, N/\mathrm{t}-M;\mathbb{Q})$, that is

$$H_i(N/\mathrm{t}, N/\mathrm{t}-M;\mathbb{Q}) \xrightarrow[\cong]{\cap\omega} H_{i-n}(M;\mathbb{Q}) \tag{4.2}$$

is an isomorphism for all i.
If n is odd, $H_i(N/\mathrm{t}, N/\mathrm{t}-M;\mathbb{Q}) = 0$ for all i.

Proof. Since the action is fibrewise and equal to $-id$ on each fibre, $N/\mathrm{t}\to M$ is also a locally trivial fibre bundle: Let N be trivial over U_i via $h_i : U_i\times\mathbb{R}^n\to N_{|U_i}$. h_i is equivariant by definition and since the action is fibrewise we get

$$\begin{tikzcd} U_i\times(\mathbb{R}^n/\sim) \arrow[rr, "h_i/\mathrm{t}", "\cong"'] \arrow[dr] & & N_{|U_i}/\sim \arrow[dl] \\ & U_i & \end{tikzcd}.$$

Note that again, we made use of the fact that U_i is a (open) manifold of finite dimension and hence locally compact, so that $id_{U_i}\times(v\mapsto[v])$ is a quotient map making the same identifications as $(x,v)\mapsto[(x,v)]$ implying that $U_i\times(\mathbb{R}^n/\sim)\cong(U_i\times\mathbb{R}^n)/\sim$. We can thus write $(x,[v])$ for points in N/t.
Let U_j be another trivializing neighbourhood. If $U_i\cap U_j\neq\emptyset$ and ϕ_{ij} denotes the transition function, then we have $h_j^{-1}\circ h_i\big(\mathrm{t}(x,v)\big) = h_j^{-1}\circ h_i\big((x,-v)\big) = \big(x,\phi_{ij}(x)(-v)\big) = \big(x,-\phi_{ij}(x)(v)\big) = \mathrm{t}\Big(h_j^{-1}\circ h_i\big((x,v)\big)\Big)$. Thus $\{(h_i/\mathrm{t}, U_i)\}_i$ is a bundle atlas for $N/\mathrm{t}\to M$.

The typical fibre of N/t is $\mathbb{R}^n/(v \sim -v) = \mathbb{R}^n/\sim$. Let $z : M \to N$ be the zero section and $q : N \to N/\mathrm{t}$, then $q \circ z : M \to N/\mathrm{t}$ embeds M as the zero in each fibre into N/t. As

$$h_i/\mathrm{t} : \Big(U_i \times (\mathbb{R}^n/\sim), U_i \times \big((\mathbb{R}^n - 0)/\sim\big)\Big) \to (N/\mathrm{t}_{|U_i}, N/\mathrm{t}_{|U_i} - U_i)$$

are maps of pairs and the diagrams

$$\begin{tikzcd} U_i \times (\mathbb{R}^n/\sim) \arrow[rr, "h_i/\mathrm{t}"] \arrow[dr] & & N/\mathrm{t}_{|U_i} \arrow[dl] \\ & U_i & \end{tikzcd}$$

commute, $(N/\mathrm{t}, N/\mathrm{t}-M) \to M$ is a fibre bundle pair. The fibre bundle $(N/\mathrm{t}, N/\mathrm{t}-M) \to M$ has typical fibre $(\mathbb{R}^n/\sim, (\mathbb{R}^n - 0)/\sim)$. Since $\mathbb{R}^n/\sim$ is still contractible, we have

$$H_i(\mathbb{R}^n/\sim, (\mathbb{R}^n - 0)/\sim) \cong \widetilde{H}_{i-1}((\mathbb{R}^n - 0)/\sim) \cong \widetilde{H}_{i-1}(S^{n-1}/\sim) = \widetilde{H}_{i-1}(\mathbb{R}P^{n-1}).$$

Here $\widetilde{H}_*$ denotes reduced homology, which satisfies $\widetilde{H}_i(\{pt\}) = 0$ for all i. If we take coefficients in a field with characteristic $\neq 2$ and n even, then $(N/\mathrm{t}, N/\mathrm{t}-M) \to M$ is a relative fibration whose fibres are homological (reduced) n-spheres, thus for $\mathbb{Q}$-coefficients

$$H_i(\mathbb{R}^n/\sim, (\mathbb{R}^n - 0)/\sim; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = n \\ 0 & \text{otherwise} \end{cases}.$$

For a simply-connected manifold M the bundle $(N/\mathrm{t}, N/\mathrm{t} - M) \to M$ is then $\mathbb{Q}$-orientable, provided that $rank(N)$, which equals $dim(M)$, is even ([**Hat02**, Section 4.D]). That is, cohomology classes in $H^*(N/\mathrm{t}, N/\mathrm{t} - M; \mathbb{Q})$, which restrict to the generators of the fibres exist. Thus with $\mathbb{Q}$-coefficients the Leray-Hirsch theorem assures the existence of a Thom isomorphism:

$$H_i(N/\mathrm{t}, N/\mathrm{t} - M; \mathbb{Q}) \xrightarrow[\cong]{\cap\omega} H_{i-n}(M; \mathbb{Q}).$$

Here $\omega \in H^n(N/\mathrm{t}, N/\mathrm{t} - M; \mathbb{Q})$ is a rational Thom class of the bundle $N/\mathrm{t} \to M$. ω in nonzero and generates $H^n(N/\mathrm{t}, N/\mathrm{t} - M; \mathbb{Q})$.
In case n is odd, we have $H_i(N/\mathrm{t}, N/\mathrm{t} - M; \mathbb{Q}) = 0$ for all $i \geq 0$ (compare [**Spa95**, Chapter 5, Excercise E1]).

The embedding s'_M/T is a map of pairs:

$$s'_M/\mathrm{T} : \big((D_\varepsilon N)/\mathrm{t}, (D_\varepsilon N)/\mathrm{t} - M\big) \to \big(M^2/\mathrm{T}, M^2/\mathrm{T} - (\Delta/\mathrm{T})(M)\big).$$

Its image is the open neighboorhood $q_\mathrm{T}(U_{M,2\varepsilon})$ (with an $\varepsilon > 0$ and w.r.t. some metric g on M) of $(\Delta/\mathrm{T})(M) \cong \Delta(M)$ inside M^2/T. On the zero section $M \subset N/\mathrm{t}$ the map s'_M/T restricts to the inclusion $M \cong (\Delta/\mathrm{T})(M) \hookrightarrow M^2/\mathrm{T}$. It thus has the properties of a tubular neighbourhood map. □

REMARK 4.3.
- Orientability of N is enough to ensure the existence of ω, simply-connectedness of the base M is not necessary (compare the remark at the end of this section).
- Interestingly, in the case $n = 2$ the fibres of $(N/\mathrm{t}, N/\mathrm{t} - M) \to M$ are homologically 2-spheres for any coefficients, since $\mathbb{R}P^1 \cong S^1$, just as if we had an actual two-dimensional vector bundle around $(\Delta/\mathrm{T})(M)$.

In fact, if $M = S^2$ we actually have that $(S^2 \times S^2)/\mathrm{T}$ is homeomorphic to $\mathbb{C}P^2$ (in fact, the n-th symmetric product of S^2 is homeomorphic to $\mathbb{C}P^n$, see e.g. [**Hat02**, Example 4K.4]). So at least topologically S^2 embeds diagonally (via Δ/T) into $\mathbb{C}P^2$ and we can ask whether it has a tubular neighbourhood.
In all other cases the quotient M^2/T does not seem to be a manifold.

Next we pull N/t back along $ev_0/\psi : \mathcal{F}/\psi \to M$ to get a bundle $ev_0/\psi^*(N/\mathrm{t}) \to \mathcal{F}/\psi$ with typical fibre again $\mathbb{R}^n/(v \sim -v)$. Via the pullback zero section $ev_0/\psi^*(q \circ z) : \mathcal{F}/\psi \to ev_0/\psi^*(N/\mathrm{t})$, $[(\gamma, \delta)] \mapsto \Big([(\gamma,\delta)], q \circ z \circ ev_0/\psi\big([(\gamma,\delta)]\big)\Big) = \big([(\gamma,\delta)], (\gamma(0), [0])\big)$, $\mathcal{F}/\psi$ embeds into $ev_0/\psi^*(N/\mathrm{t})$. As before, we wish to have a map $s'_{\mathcal{F}/\psi}$ that makes the diagram

$$\begin{tikzcd} ev_0/\psi^*\big((D_\varepsilon N)/\mathrm{t}\big) \arrow[r, dashed, "s'_{\mathcal{F}/\psi}"] \arrow[d] & (\Lambda \times \Lambda)/\psi \arrow[d, "(ev_0 \times ev_0)/\psi"] \\ (D_\varepsilon N)/\mathrm{t} \arrow[r, hook, "s'_M/\mathrm{T}"'] & M \times M/\mathrm{T} \end{tikzcd}$$

commutative.
Unfortunately such a map does not seem to exist since already too many identifications have been made in $ev_0/\psi^*\big((D_\varepsilon N)/\mathrm{t}\big)$: If we consider the map $Q = q_{\mathcal{F}} \times q_N : {ev_0}^*(D_\varepsilon N) \to ev_0/\psi^*\big((D_\varepsilon N)/\mathrm{t}\big)$ with $q_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}/\psi$ and $q_N : N \to N/\mathrm{t}$ the quotient maps, then Q is the identification map of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action. More precisely, Q is the quotient map of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action induced by the involutions $\psi \times id_N, id_{\mathcal{F}} \times \mathrm{t}$ and $\psi \times \mathrm{t}$. This holds since Q is a quotient map: Since $\mathbb{Z}_2$ is compact and $\mathcal{F}$ and N are Hausdorff, the maps $q_{\mathcal{F}}$ and q_N are closed, continuous, surjective and the preimages of points are compact. That is $q_{\mathcal{F}}$ and q_N are surjective and proper and continuous, i.e. $q_{\mathcal{F}}$ and q_N are so called perfect maps. If follows that also $Q = q_{\mathcal{F}} \times q_N$ is a perfect map (see e.g. [**LS15**, Folgerung 4.12(Produktsatz)]). So Q is a closed, continuous and surjective map and hence a quotient map (see e.g. [**Lee11**, Proposition 3.69]).

Let us instead look at the pullback square

$$\begin{tikzcd} {ev_0}^*(D_\varepsilon N) \arrow[r, "pr_N"] \arrow[d, "pr_{\mathcal{F}}"'] & D_\varepsilon N \arrow[d, "p"] \\ \mathcal{F} \arrow[r, "ev_0"'] & M \end{tikzcd}$$

once more. The evaluation ev_0 is $\mathbb{Z}_2$-equivariant with respect to the actions ψ on $\mathcal{F}$ and the trivial action on M:

$$(ev_0 \circ \psi)\big((\gamma,\delta)\big) = \bar{\delta}(0) = \bar{\gamma}(0)) = \gamma(0) = \delta(0) = ev_0\big((\gamma,\delta)\big).$$

The projection p has the same property with ψ replaced by t:

$$p \circ \mathrm{t}(x, v) = p(x, -v) = x = p(x, v).$$

It follows that the maps pr_N and $pr_{\mathcal{F}}$ are $\mathbb{Z}_2$-equivariant with respect to the diagonal $\mathbb{Z}_2$-action on ${ev_0}^*(N)$ induced by the involution

$$\psi \times \mathrm{t} : {ev_0}^*(D_\varepsilon N) \to {ev_0}^*(D_\varepsilon N)$$

and those induced by the involutions $\mathrm{t} : N \to N$ and $\psi : \mathcal{F} \to \mathcal{F}$ respectively:

$$\begin{aligned}(pr_N \circ \psi \times \mathrm{t})\big((\gamma,\delta),(x,v)\big) &= pr_N\big((\bar\delta,\bar\gamma),(x,-v)\big) = (x,-v) = \mathrm{t}\big((x,v)\big)\\ &= (\mathrm{t} \circ pr_N)\big((\gamma,\delta),(x,v)\big)\end{aligned}$$

and

$$\begin{aligned}(pr_{\mathcal{F}} \circ \psi \times \mathrm{t})\big((\gamma,\delta),(x,v)\big) &= pr_{\mathcal{F}}\big((\bar\delta,\bar\gamma),(x,-v)\big) = (\bar\delta,\bar\gamma) = \psi(\gamma,\delta)\\ &= (\psi \circ pr_{\mathcal{F}})\big((\gamma,\delta),(x,v)\big).\end{aligned}$$

We thus get the following commutative diagram:

$$\begin{tikzcd}
{ev_0}^*(D_\varepsilon N) \arrow[rrr, "pr_N"] \arrow[drr, "q_{\psi\times\mathrm{t}}"] \arrow[dd, "pr_{\mathcal{F}}"'] & & & D_\varepsilon N \arrow[dr, "q_N"] \arrow[dd, "p"] & \\
& & \big({ev_0}^*(D_\varepsilon N)\big)/(\psi\times\mathrm{t}) \arrow[rr, "pr_N/(\psi\times\mathrm{t})"] \arrow[dd, "pr_{\mathcal{F}}/(\psi\times\mathrm{t})"'] & & (D_\varepsilon N)/\mathrm{t} \arrow[dd, "p/\mathrm{t}"] \\
\mathcal{F} \arrow[rrr, "ev_0"'] \arrow[drr, "q_{\mathcal{F}}"'] & & & M \arrow[dr, "="'] & \\
& & \mathcal{F}/\psi \arrow[rr, "ev_0/\psi"'] & & M
\end{tikzcd} \tag{4.3}$$

Note that, unfortunately, the commuting diagram of quotients is not a pullback and the map ${ev_0}^*(D_\varepsilon N)/(\psi \times \mathrm{T}) \to \mathcal{F}/\psi$ is not a disk bundle. It is not even a bundle with fixed type of fibre:

- If a pair of curves $(\gamma,\delta) \in \mathcal{F}$ is not fixed under $\psi = \mathrm{T} \circ (\vartheta \times \vartheta)$, then no identifications are made in the fibre ${ev_0}^*(N)_{(\gamma,\delta)}$ over (γ,δ).
- If instead $(\gamma,\delta) = (\bar\delta,\bar\gamma) = \psi(\gamma,\delta)$, then the points $\big((\gamma,\delta),(x,v)\big)$ and $\big((\bar\delta,\bar\gamma),(x,-v)\big) = \big((\gamma,\delta),(x,-v)\big)$, which both lie in ${ev_0}^*(N)_{(\gamma,\delta)}$, are identified.

Note also that the zero section $\mathcal{F} \to {ev_0}^*(D_\varepsilon N),\ (\gamma,\delta) \mapsto \big((\gamma,\delta),(\gamma(0),0)\big) = \big((\gamma,\delta),(ev_0(\gamma,\delta),0)\big)$ is equivariant so that $\mathcal{F}/\psi \to {ev_0}^*(D_\varepsilon N)/(\psi \times \mathrm{t}),\ [(\gamma,\delta)] \mapsto \big[\big((\gamma,\delta),(\gamma(0),0)\big)\big]$ embeds $\mathcal{F}/\psi$ into $\to {ev_0}^*(D_\varepsilon N)/(\psi \times \mathrm{t})$.

Nevertheless, we have

Proposition 4.4. *Let (M,g) be a compact Riemannian manifold and let $\Lambda = \Lambda M$ denote its free loop space. Then, the tubular neighbourhood map $s_{\mathcal{F}} : {ev_0}^*(N) \to \Lambda \times \Lambda$ of Section 2 of Chapter 2 is $\mathbb{Z}_2$-equivariant with respect to the actions $\psi \times \mathrm{t} = (\mathrm{T} \circ \vartheta \times \vartheta) \times \mathrm{t}$ on ${ev_0}^*(N)$ and $\psi = \mathrm{T} \circ \vartheta \times \vartheta$ on Λ^2. It descends to a open embedding $s_{\mathcal{F}}/\psi$ of ${ev_0}^*(N)/(\psi \times \mathrm{t})$ into Λ^2/ψ with image $U_{\mathcal{F},2\varepsilon}/\psi = \{(\gamma,\delta) \in \Lambda^2 |\ d_g\big(\gamma(0),\delta(0)\big) < 2\varepsilon\}/\psi$ for some $\varepsilon > 0$ that restricts to the inclusion on $\mathcal{F}/\psi$.*
Moreover, the diagram

$$\begin{tikzcd}
{ev_0}^*(N)/(\psi \times \mathrm{t}) \arrow[r, hook, "s_{\mathcal{F}}/\psi"] \arrow[d, "(pr_N/\psi\times\mathrm{t})"'] & (\Lambda \times \Lambda)/\psi \arrow[d, "(ev_0 \times ev_0)/\psi"] \\
N/\mathrm{t} \arrow[r, hook, "s_M/\mathrm{T}"] & (M \times M)/\mathrm{T}
\end{tikzcd} \tag{4.4}$$

is commutative and $(ev_0 \times ev_0)/\psi(U_{\mathcal{F},2\varepsilon}/\psi) = U_{M,2\varepsilon}/\mathrm{T}$.

Proof. Recall the symmetrized tubular neighbourhood $s'_{\mathcal{F}} : {ev_0}^*(D_\varepsilon N) \to \Lambda^2$ of $\mathcal{F}$ from Section 2 of Chapter 2:

$$s'_{\mathcal{F}}(\gamma,\delta,v) := \big(\lambda(\gamma,-v), \lambda(\delta,v)\big).$$

Since

$$\begin{tikzcd}
\big((\gamma,\delta),(x,v)\big) \arrow[r, mapsto, "s'_{\mathcal{F}}"] \arrow[d, mapsto, "\psi\times \mathrm{t}"] & \big(\lambda(\gamma,-v),\lambda(\delta,v)\big) \arrow[d, mapsto, "\psi"] \\
\big((\bar\delta,\bar\gamma),(x,-v)\big) \arrow[r, mapsto, "s'_{\mathcal{F}}"] & \big(\lambda(\bar\delta,v),\lambda(\bar\gamma,-v)\big)
\end{tikzcd}$$

commutes, $s'_{\mathcal{F}}$ is equivariant and descends to the quotients to give a map

$$\begin{aligned} s'_{\mathcal{F}}/\psi : & {ev_0}^*(D_\varepsilon N)/(\psi\times \mathrm{t}) \to \Lambda^2/\psi \\ & \big[\big((\gamma,\delta),(x,v)\big)\big] \mapsto \big[\big(\lambda(\gamma,-v),\lambda(\delta,v)\big)\big]. \end{aligned}$$

We now show that $s'_{\mathcal{F}}/\psi$ has most features of a tubular neighbourhood map:

- The commutative diagram

$$\begin{tikzcd}
{ev_0}^*(D_\varepsilon N) \arrow[r, "s'_{\mathcal{F}}"] \arrow[d] & \Lambda^2 \arrow[d] \\
{ev_0}^*(D_\varepsilon N)/(\psi\times \mathrm{t}) \arrow[r, "s'_{\mathcal{F}}/\psi"] & \Lambda^2/\psi
\end{tikzcd}$$

  shows that $s'_{\mathcal{F}}/\psi$ is an open map with image $U_{\mathcal{F},2\varepsilon}/\psi$. (Note that $U_{\mathcal{F},2\varepsilon} = \{(\gamma,\delta) \in \Lambda^2 |\ d_g\big(\gamma(0),\delta(0)\big) < 2\varepsilon\}$ is invariant under ψ).
- As $s'_{\mathcal{F}}/\psi\Big(\big[\big((\gamma,\delta),(x,0)\big)\big]\Big) = \big[\big(\lambda(\gamma,-0),\lambda(\delta,0)\big)\big] = [(\gamma,\delta)]$, we see that it restricts to the inclusion $i_{\mathcal{F}/\psi} : \mathcal{F}/\psi \hookrightarrow \Lambda^2/\psi$ on the zero section of the bundle ${ev_0}^*(D_\varepsilon N)/(\psi\times \mathrm{T}) \to \mathcal{F}/\psi$.
- Is is also easily seen that the inverse μ (see Section 2 of Chapter 2) of $s'_{\mathcal{F}}$ is also equivariant, hence μ/ψ is an inverse.

An open injective continuous map is an embedding. So $s'_{\mathcal{F}}/\psi : {ev_0}^*(D_\varepsilon N)/(\psi\times \mathrm{t}) \to \Lambda^2/\psi$ is an embedding with image an open neighbourhood of $\mathcal{F}/\psi$. Furthermore, the diagram

$$\begin{tikzcd}
{ev_0}^*(D_\varepsilon N)/(\psi\times \mathrm{t}) \arrow[r, "s'_{\mathcal{F}}/\psi"] \arrow[d, "pr_N/(\psi\times \mathrm{t})"'] & (\Lambda^2)/\psi \arrow[d, "(ev_0\times ev_0)/\psi"] \\
(D_\varepsilon N)/\mathrm{t} \arrow[r, hook, "s'_M/\mathrm{T}"'] & M^2/\mathrm{T}
\end{tikzcd}$$

is commutative as

$$\begin{tikzcd}
\big[\big((\gamma,\delta),(x,v)\big)\big] \arrow[r, mapsto, "s'_{\mathcal{F}}/\psi"] \arrow[d, mapsto, "pr_N/(\psi\times \mathrm{t})"'] & \big[\big(\lambda(\gamma,-v),\lambda(\delta,v)\big)\big] \arrow[d, mapsto, "(ev_0\times ev_0)/\psi"] \\
{[(x,v)]} \arrow[r, mapsto, "s'_M/\mathrm{T}"] & \big[\big(\exp(\gamma(0),-v),\exp(\delta(0),v)\big)\big]
\end{tikzcd}$$

shows. It follows that the diagram

$$\begin{tikzcd}
\big(ev_0{}^*(D_\varepsilon N), ev_0{}^*(D_\varepsilon N) - \mathcal{F}\big) \arrow[r, hook, "s'_{\mathcal{F}}", "\cong"'] \arrow[d] & (U_{\mathcal{F},2\varepsilon}, U_{\mathcal{F},2\varepsilon} - \mathcal{F}) \arrow[d] \\
\big(ev_0{}^*(D_\varepsilon N)/(\psi \times \mathrm{t}), ev_0{}^*(D_\varepsilon N)/(\psi \times \mathrm{t}) - \mathcal{F}/\psi\big) \arrow[r, hook, "s'_{\mathcal{F}}/\psi", "\cong"'] \arrow[d] & (U_{\mathcal{F},2\varepsilon}/\vartheta, U_{\mathcal{F},2\varepsilon}/\psi - \mathcal{F}/\psi) \arrow[d] \\
\big((D_\varepsilon N)/\mathrm{t}, (D_\varepsilon N)/\mathrm{t} - M\big) \arrow[r, hook, "s'_M/\mathrm{T}", "\cong"'] & (U_{M,2\varepsilon}/\mathrm{T}, U_{M,2\varepsilon}/\mathrm{T} - \Delta(M))
\end{tikzcd}$$

is commutative. Let us now, as before, precompose these maps with a shrinking of the fibre ζ_ε as we did in Section 2 of Chapter 2:

- $\zeta_\varepsilon : N \to D_\varepsilon N,\ (p, v) \mapsto (p, \frac{\varepsilon v}{1+|v|})$ descends to a fibrewise homeomorphism $N/\mathrm{t} \cong (D_\varepsilon N)/\mathrm{t}$ over M.
- $\zeta_\varepsilon : ev_0{}^*(N) \to D_\varepsilon ev_0{}^*(N),\ \big((\gamma, \delta), (x, v)\big) \mapsto \big((\gamma, \delta), (x, \frac{\varepsilon v}{1+|v|})\big)$ gives $ev_0{}^*(N) \cong D_\varepsilon ev_0{}^*(N) = ev_0{}^*(D_\varepsilon N)$ fibre-preservingly over $\mathcal{F}$. As

$$\big((\gamma, \delta), (x, v)\big) \mapsto \big((\gamma, \delta), (x, \tfrac{\varepsilon v}{1+|v|})\big) \mapsto \big((\bar\delta, \bar\gamma), (x, -\tfrac{\varepsilon v}{1+|v|})\big) \leftarrow\!\shortmid \big((\bar\delta, \bar\gamma), (x, -v)\big) \leftarrow\!\shortmid \big((\gamma, \delta), (x, v)\big),$$

  we get a fibrewise homeomorphism $ev_0{}^*(N)/(\psi \times \mathrm{t}) \cong ev_0{}^*(D_\varepsilon N)/(\psi \times \mathrm{t})$ over $\mathcal{F}/\psi$.
- Also all the projections certainly commute with these shrinking maps, so that the diagram 4.3 still commutes if we remove the all the D_ε prefixes.

We can thus define, as before, the map $s_{\mathcal{F}}/\psi : ev_0{}^*(N)/(\psi \times \mathrm{t}) \to \Lambda^2/\psi$ which has the desired properties. $\square$

The bundles involved here are $N/\mathrm{t} \to M$ (fibre bundle), $ev_0{}^*(N)/(\psi \times \mathrm{t}) \to \mathcal{F}/\psi$ (no typical fibre) and $ev_0{}^*(N) \to \mathcal{F}$ (vector bundle). We would like these bundle projections to induce isomorphism on homology. Let us therefore at this point check whether these bundle projections are homotopy equivalences:

- The zero section of $p/\mathrm{t} : N/\mathrm{t} \to M$ is obviously given by $q_N \circ z$ where z is the zero section of $p : N \to M$: $p/\mathrm{t} \circ q_N \circ z = p \circ z = id_M$.
  The fibrewise straight line homotopy $H : N \times [0, 1] \to N,\ H\big((x, v), t\big) = (x, tv)$ gives $z \circ p \simeq id_N$. With the trivial action on $[0, 1]$ we get a homotopy $H/\mathrm{t} : N/\mathrm{t} \times [0, 1] \to N/\mathrm{t},\ \big((x, [v]), t\big) \mapsto (x, [tv])$ since $H\big(\mathrm{t} \times id\big((x, v), t\big)\big) = H\big((x, -v), t\big) = (x, -tv) = \mathrm{t} \circ H\big((x, v), t\big)$ (see section 1 of chapter 4). As $z(M) = \{(x, 0)\} \subset N$ is an invariant subspace, H/t is a homotopy between $z/\mathrm{t} \circ p/\mathrm{t}$ and $id_{N/\mathrm{t}}$.
- For the bundle $pr_{\mathcal{F}}/(\psi \times \mathrm{t}) : ev_0{}^*(N)/(\psi \times \mathrm{t}) \to \mathcal{F}/\psi$ the proof is the same.

Let now $a \in H^n\big(N/\mathrm{t}, N/\mathrm{t} - M; R\big)$ be any class. Then we get the following commutative diagram by naturality of the cap product:

$$
\begin{tikzcd}
H_{i-n}(N_{\mathcal{F}}; R) \arrow[d, "g_*"'] & & H_i\big(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}; R\big) \arrow[ll, "\cap g^*\Big((q_{\psi\times\mathrm{t}})^*\big((pr_N/(\psi\times\mathrm{t}))^*(a)\big)\Big)"'] \arrow[d, "g_*"] \\
H_{i-n}\big(ev_0{}^*(N); R\big) \arrow[d, "(q_{\psi\times\mathrm{t}})_*"'] & & H_i\big(ev_0{}^*(N), ev_0{}^*(N) - \mathcal{F}; R\big) \arrow[ll, "\cap(q_{\psi\times\mathrm{t}})^*\big((pr_N/(\psi\times\mathrm{t}))^*(a)\big)"'] \arrow[d, "(q_{\psi\times\mathrm{t}})_*"] \\
H_{i-n}\big(ev_0{}^*(N)/(\psi\times\mathrm{t}); R\big) \arrow[d, "\big(pr_N/(\psi\times\mathrm{t})\big)_*"'] & & H_i\big(ev_0{}^*(N)/(\psi\times\mathrm{t}), ev_0{}^*(N)/(\psi\times\mathrm{t}) - \mathcal{F}/\psi; R\big) \arrow[ll, "\cap\big(pr_N/(\psi\times\mathrm{t})\big)^*(a)"'] \arrow[d, "\big(pr_N/(\psi\times\mathrm{t})\big)_*"] \\
H_{i-n}\big(N/\mathrm{t}; R\big) & & H_i\big(N/\mathrm{t}, N/\mathrm{t} - M; R\big). \arrow[ll, "\cap a"']
\end{tikzcd}
$$

Here $g : N_{\mathcal{F}} \to ev_0{}^*(N)$ is the isomorphism of bundles over $\mathcal{F}$.
Note that if there is a class a with the property that $g^*\Big((q_{\vartheta\times\mathrm{T}})^*\big((pr_N/(\psi\times\mathrm{t}))^*(a)\big)\Big)$ equals the Thom class $\tau_{\mathcal{F}}$ of the bundle $N_{\mathcal{F}} \to \mathcal{F}$ then the two horizontal arrows at the top are isomorphisms.

Definition 4.5. Let M be a compact, connected, oriented smooth n-manifold and let $\Lambda = \Lambda M$ denote its free loop space. Let $a \in H^k\big(N/\mathrm{t}, N/\mathrm{t} - M; R\big)$ be some cohomology class with coefficient in a commutative unital ring R. We set

$$\hat{a} := \big(pr_N/(\psi\times\mathrm{t})\big)^*(a) \in H^k\big(ev_0{}^*(N)/(\psi\times\mathrm{t}), ev_0{}^*(N)/(\psi\times\mathrm{t}) - \mathcal{F}/\psi; R\big)$$

and on the homology $H_*(\Lambda/\vartheta; R)$ we define the product

$$A^a_\vartheta : H_i(\Lambda/\vartheta; R) \times H_j(\Lambda/\vartheta; R) \to H_{i+j-k}(\Lambda/\vartheta; R) \tag{4.5}$$

to be the composition down the middle of the following commutative diagram.

(4.6)

$$
\begin{tikzcd}
 & A^a_\vartheta \arrow[d] & \\
 & H_i(\Lambda/\vartheta) \times H_j(\Lambda/\vartheta) \arrow[r, dashed] \arrow[dl, "tr \times tr"'] \arrow[dd, "q_* \circ \times \circ tr \times tr"] & H_i(M) \times H_j(M) \arrow[d, "\times"] \\
H_i(\Lambda) \times H_j(\Lambda) \arrow[d, "\times"'] \arrow[ddddddd, bend right, "\diamond^a"'] & & H_i(M^2) \arrow[d, "q_*"] \\
H_{i+j}(\Lambda^2) \arrow[r, "q_*"] \arrow[d] & H_{i+j}(\Lambda^2/\psi) \arrow[r, "(ev_0 \times ev_0/\psi)_*"] \arrow[d] & H_{i+j}(M^2/\mathrm{T}) \arrow[d] \\
H_{i+j}(\Lambda^2, \Lambda^2 - \mathcal{F}) \arrow[r, "q_*"] \arrow[d, "\cong"'] & H_{i+j}(\Lambda^2/\psi, \Lambda^2/\psi - \mathcal{F}/\psi) \arrow[r, "(ev_0 \times ev_0/\psi)_*"] \arrow[d, "\cong"'] & H_{i+j}\big(M^2/\mathrm{T}, M^2/\mathrm{T} - \Delta(M)\big) \arrow[d, "\cong"'] \\
H_{i+j}(U_{\mathcal{F},\varepsilon}, U_{\mathcal{F},\varepsilon} - \mathcal{F}) \arrow[r, "q_*"] \arrow[d, "\cong"', "(s_{\mathcal{F}*})^{-1}"] & H_{i+j}(U_{\mathcal{F},2\varepsilon}/\psi, U_{\mathcal{F},2\varepsilon}/\psi - \mathcal{F}/\psi) \arrow[r, "(ev_0 \times ev_0/\psi)_*"] \arrow[d, "\cong"', "\big((s_{\mathcal{F}}/\psi)_*\big)^{-1}"] & H_{i+j}\big(U_{M,2\varepsilon}/\mathrm{T}, U_{M,2\varepsilon}/\mathrm{T} - \Delta(M)\big) \arrow[d, "\cong"', "(s_M/\mathrm{T}_*)^{-1}"] \\
H_{i+j}(N_{\mathcal{F}}, N_{\mathcal{F}} - \mathcal{F}) \arrow[r, "(q_{\psi \times \mathrm{t}} \circ g)_*"'] \arrow[d, "\cap (q_{\psi \times \mathrm{t}} \circ g)^*(\hat{a})"] & H_{i+j}\big(\frac{ev_0{}^*(N)}{\psi \times \mathrm{t}}, \frac{ev_0{}^*(N)}{\psi \times \mathrm{t}} - \mathcal{F}/\psi\big) \arrow[r, "(pr_N/(\psi \times \mathrm{t}))_*"'] \arrow[d, "\cap \hat{a}"] & H_{i+j}(N/\mathrm{t}, N/\mathrm{t} - M) \arrow[d, "\cap a"] \\
H_{i+j-n}(N_{\mathcal{F}}) \arrow[r, "(q_{\psi \times \mathrm{t}} \circ g)_*"'] \arrow[d, "\cong"'] & H_{i+j-n}\big(ev_0{}^*(N)/(\psi \times \mathrm{t})\big) \arrow[r, "(pr_N/(\psi \times \mathrm{t}))_*"'] \arrow[d, "\cong"'] & H_{i+j-n}(N/\mathrm{t}) \arrow[d, "\cong"'] \\
H_{i+j-k}(\mathcal{F}) \arrow[r, "q_*"'] \arrow[d, "\phi_*"'] & H_{i+j-k}(\mathcal{F}/\psi) \arrow[r, "ev_0/\psi_*"'] \arrow[d, "\phi/\vartheta_*"'] & H_{i+j-k}(M) \\
H_{i+j-k}(\Lambda) \arrow[r, "q_*"'] & H_{i+j-k}(\Lambda/\vartheta) \arrow[ur, "ev_0/\vartheta_*"'] &
\end{tikzcd}
$$

where $tr : H_i(\Lambda/\vartheta) \to H_i(\Lambda)$ is the transfer map and the map ϕ/ϑ is concatenation which we defined in the next section.

We observe that

- if we denote the composition down the left by $\diamond^a$, then we have
$$A^a_\vartheta(a,b) = q_*\big(tr(a)\diamond^a tr(b)\big).$$
So A^a_ϑ has the form of a transfer product, but it is only equal to P_ϑ when there is a class $a \in H^n\big(N/\mathrm{t}, N/\mathrm{t}-M;R\big)$ with $(q_{\psi\times\mathrm{t}}\circ g)^*(\hat{a}) = \tau_{\mathcal{F}}$. This also implies that A^a_ϑ is independent of the metric g on M chosen to define $s_{\mathcal{F}}/\psi$.
- the existence of the dashed arrow might depend on the coefficients. For $\mathbb{Q}$ as coefficient ring, $\frac{1}{4}(ev_0/\vartheta_* \times ev_0/\vartheta_*)$ works. This is interesting, since here we have a relation to some product on M as $ev_0 : \Lambda \to M$ is equivariant because the base point is fixed under ϑ. For even-dimensional spheres and rational homology the composition down the right in the above diagram seems to be the intersection product if $k = n = dim(M)$ and $a = \omega$, where $\omega \in H^n(N/\mathrm{t}, N/\mathrm{t}-M;\mathbb{Q})$ (compare discussion below).

Let us now investigate under which conditions we can have $(q_{\vartheta\times\mathrm{T}}\circ g)^*(\hat{a}) = \tau_{\mathcal{F}}$: The bundles involved fit into the commutative diagram

$$\begin{tikzcd}
 & & & & N \arrow[d, "q_N"] \\
N_{\mathcal{F}} \arrow[r, "g", "\cong"'] \arrow[ddd] & ev_0^*(N) \arrow[urrr, "pr_N"] \arrow[r, "q_{\psi\times\mathrm{t}}"] \arrow[drr, "Q=q_{\mathcal{F}}\times q_N"'] \arrow[ddd] & ev_0{}^*(N)/(\psi\times\mathrm{t}) \arrow[rr, "pr_N/(\psi\times\mathrm{t})"] \arrow[dr, "a"] \arrow[ddd] & & N/\mathrm{t} \arrow[ddd] \\
 & & & ev_0/\psi^*(N/\mathrm{t}) \arrow[ur, "b"] \arrow[ddl] & \\
 & & & & \\
\mathcal{F} \arrow[r, "="'] & \mathcal{F} \arrow[r, "q_{\mathcal{F}}"'] & \mathcal{F}/\psi \arrow[rr, "ev_0/\psi"'] & & M.
\end{tikzcd}$$

We recall that we have $\tau_{\mathcal{F}} = g^*\big(pr_N{}^*(\tau_M)\big)$ where $\tau_M \in H^n(N, N-M;R)$ is the Thom class of the vector bundle $N \to M$. For rational coefficients we have found a Thom class $\omega \in H^n(N/\mathrm{t}, N/\mathrm{t}-M;\mathbb{Q})$ if n is even (see equation (4.2)).

Lemma 4.6. *Let M be a simply-connected, compact manifold with $dim(M) = n$ even. Let $\tau_M \in H^n(N, N-M;\mathbb{Q})$ be the Thom class of the diagonal inclusion $\Delta : M \hookrightarrow M^2$ for which $\tau_{\mathcal{F}} = g^*\big(pr_N{}^*(\tau_M)\big)$ holds. Let $\omega \in H^n(N/\mathrm{t}, N/\mathrm{t}-M;\mathbb{Q})$ denote the rational Thom class of $N/\mathrm{t} \to M$ from Lemma 4.2.*
Then the following holds:

(1) $q_N{}^*(\omega) = \tau_M$,
(2) $\hat{\omega} := \big(pr_N/(\psi\times\mathrm{t})\big)^*(\omega)$ is nonzero.

Proof. This follows from the properties of transfer maps (see Section 2) and the fact that Thom classes pull back to Thom classes:

(1) $q_N = q : N \to N/\mathrm{t}$ is a ramified covering being the orbit map of a $\mathbb{Z}_2$-action. Also $N, N/\mathrm{t}, (N-M)$ and $(N/\mathrm{t}-M)$ have the homotopy type of CW-complexes ([**FP90**, Theorem 5.4.2]). It follows that $q_{N*} = q_* : H_n(N;\mathbb{Q}) \to H_n(N/\mathrm{t};\mathbb{Q})$ and $q_* : H_n(N-M;\mathbb{Q}) \to H_n(N/\mathrm{t}-M;\mathbb{Q})$ are surjective, the latter is even an actual covering map. Now, as $Ext_{\mathbb{Z}}(A,\mathbb{Q}) = 0$ for any abelian group A as $\mathbb{Q}$ is divisible

and hence an injective $\mathbb{Z}$-module ([**Bre93**, Chapter V, Definition 6.1 + Proposition 6.2 + Proposition 6.6 (2)]). It follows, by naturality of the exact sequences in the universal coefficient theorem ([**Bre93**, Chapter V, Theorem 7.1]), that the diagram

$$\begin{tikzcd} H^n(N;\mathbb{Q}) \arrow[r, "\cong"] & Hom_{\mathbb{Z}}\big(H_n(N;\mathbb{Z}),\mathbb{Q}\big) \\ H^n(N/\mathrm{t};\mathbb{Q}) \arrow[r, "\cong"'] \arrow[u, "q^*"] & Hom_{\mathbb{Z}}\big(H_n(N/\mathrm{t};\mathbb{Z}),\mathbb{Q}\big) \arrow[u, "(q_*)^*"'] \end{tikzcd}$$

commutes. Here $(q_*)^*$ is the map dual to q_*. For $f,g \in Hom_{\mathbb{Z}}\big(H_n(N/\mathrm{t};\mathbb{Z}),\mathbb{Q}\big)$ the equality $(q_*)^*(f) = f\circ q_* = g\circ q_* = (q_*)^*(g)$ implies that f and g agree on the image of q_*, surjectivity of q_* certainly implies injectivity of $(q_*)^*$. The diagram then shows that also q^* is injective. Likewise for $q^* : H^n(N/\mathrm{t} - M;\mathbb{Q}) \to H^n(N - M;\mathbb{Q})$.
In fact, since the bundle projection $p : N \to M$ is equivariant the induced map p/t induces an isomorphism on cohomology and so $q^* : H^k(N/\mathrm{t};\mathbb{Q}) \to H^k(N;\mathbb{Q})$ is even an isomorphism for every k. We thus have

$$\begin{tikzcd} H^{n-1}(N;\mathbb{Q}) \arrow[r] & H^{n-1}(N-M;\mathbb{Q}) \arrow[r] & H^n(N,N-M;\mathbb{Q}) \arrow[r] & H^n(N;\mathbb{Q}) \\ H^{n-1}(N/\mathrm{t};\mathbb{Q}) \arrow[r] \arrow[u, "\cong"', "q^*"] & H^{n-1}(N/\mathrm{t}-M;\mathbb{Q}) \arrow[r] \arrow[u, hook, "q^*"] & H^n(N/\mathrm{t},N/\mathrm{t}-M;\mathbb{Q}) \arrow[r] \arrow[u, "{q_N}^*"] & H^n(N/\mathrm{t};\mathbb{Q}) \arrow[u, "\cong"', "q^*"] \end{tikzcd}$$

and it follows that ${q_N}^*$ is injective. Hence ${q_N}^*(\omega)$ is nonzero if $\omega \neq 0$. We rescale (coefficients are rational) if necessary, such that ${q_N}^*(\omega) = \tau_M$.

(2) Clearly ${pr_N}^*(\tau_M) \neq 0$ since the pullback of a Thom class is a Thom class ([**AGP02**, Proposition 11.7.11]). From

$${pr_N}^*(\tau_M) = {pr_N}^*\big({q_N}^*(\omega)\big) = (q_{\psi\times\mathrm{t}})^*\big((pr_N/(\psi\times\mathrm{t}))^*(\omega)\big) = (q_{\psi\times\mathrm{t}})^*(\hat{\omega}),$$

we deduce that also $\hat{\omega} \neq 0$ holds.

□

COROLLARY 4.7. *In rational homology, ω and hence also $\hat{\omega}$ pull back to $\tau_{\mathcal{F}}$.*

PROOF. Since ${q_N}^*(\omega) = \tau_M$ we have

$$\begin{aligned} \tau_{\mathcal{F}} &= g^*\big({pr_N}^*(\tau_M)\big) = g^*\Big({pr_N}^*\big({q_N}^*(\omega)\big)\Big) \\ &= g^*\big((q_N\circ pr_N)^*(\omega)\big) = g^*\big((pr_N/(\psi\times\mathrm{t})\circ q_{\psi\times\mathrm{t}})^*(\omega)\big) \\ &= g^*\Big({q_{\psi\times\mathrm{t}}}^*\big((pr_N/(\psi\times\mathrm{t}))^*(\omega)\big)\Big) = g^*\big((q_{\psi\times\mathrm{t}})^*(\hat{\omega})\big). \end{aligned}$$

□

We make the following definition

DEFINITION 4.8. Let M be a simply-connected, compact manifold of dimension n. On $H_*(\Lambda M/\vartheta;\mathbb{Q})$ we define the following products

- for n odd, $A_\vartheta : H_i(\Lambda M/\vartheta;\mathbb{Q}) \times H_j(\Lambda M/\vartheta;\mathbb{Q}) \to H_{i+j-n}(\Lambda M/\vartheta;\mathbb{Q})$ is the zero map for all $i,j \in \mathbb{N}_0$.

- for n even, $A_\vartheta : H_i(\Lambda M/\vartheta;\mathbb{Q}) \times H_j(\Lambda M/\vartheta;\mathbb{Q}) \to H_{i+j-n}(\Lambda M/\vartheta;\mathbb{Q})$ is defined to be A^ω_ϑ.

Theorem 4.9. *Let M be a simply-connected, compact manifold of even dimension n. Then the product*

$$A_\vartheta : H_i(\Lambda M/\vartheta;\mathbb{Q}) \times H_j(\Lambda M/\vartheta;\mathbb{Q}) \to H_{i+j-n}(\Lambda M/\vartheta;\mathbb{Q})$$

is equal to the rational transfer product P_ϑ: for all $a, b \in H_*(\Lambda/\vartheta;\mathbb{Q})$

$$A_\vartheta(a,b) = q_*\big(tr(a) * tr(b)\big) = P_\vartheta(a,b) \tag{4.7}$$

holds.

Proof. The above big commutative diagram (4.6) with the class a replaced by ω and rational with coefficients and a slight abuse of notation is

$$\begin{tikzcd}
H_i(\Lambda/\vartheta) \times H_j(\Lambda/\vartheta) \arrow[d, "\text{transfer}\times\text{transfer}"] \arrow[dddddd, bend right=60, "A_\vartheta"'] & \\
H_i(\Lambda) \times H_j(\Lambda) \arrow[d, "\times"] & \\
H_{i+j}(\Lambda^2) \arrow[d, "q_*"] \arrow[dr, "id"] & \\
H_{i+j}(\Lambda^2/\psi) \arrow[d] & H_{i+j}(\Lambda^2) \arrow[l, "q_*"'] \arrow[d] \\
H_{i+j}(\Lambda^2/\psi, \Lambda^2/\psi - \mathcal{F}/\psi) \arrow[d, "\cap\hat{\omega}"] & H_{i+j}(\Lambda^2, \Lambda^2 - \mathcal{F}) \arrow[l, "q_*"'] \arrow[d, "\cap\tau_{\mathcal{F}}"] \\
H_{i+j-n}(\mathcal{F}/\psi) \arrow[d, "\phi/\psi_*\ (\text{see } 4.0.2)"] & H_{i+j-n}(\mathcal{F}) \arrow[l, "q_*"'] \arrow[d, "\phi_*"'] \\
H_{i+j-n}(\Lambda/\vartheta) & H_{i+j-n}(\Lambda) \arrow[l, "q_*"']
\end{tikzcd}$$

□

Remark 4.10. Similar to equation (3.7) we have that

$$(\psi \times \mathrm{t})^*(\tau_{\mathcal{F}}) = (-1)^n \tau_{\mathcal{F}}.$$

This is simply due to the fact that $\mathrm{t}_x = -id : N_x \to N_x$ reverses orientation if and only if $dim(N_x) = n$ is odd. It is thus not surprising that the Thom class $\tau_{\mathcal{F}}$ passes to the quotient if and only if n is even.

4.0.2. *concatenation.* The action ϑ maps $\gamma \cdot \delta$ to $\overline{\gamma \cdot \delta}$ which equals $\bar{\delta} \cdot \bar{\gamma}$. Hence concatenation at $s = \frac{1}{2}$ is equivariant with respect to ψ and ϑ:

$$\begin{tikzcd}
(\gamma, \delta) \arrow[r, mapsto] \arrow[d, mapsto, "\psi"'] & \gamma \cdot \delta \arrow[d, mapsto, "\vartheta"] \\
(\bar{\delta}, \bar{\gamma}) \arrow[r, mapsto] & \bar{\delta} \cdot \bar{\gamma}
\end{tikzcd}$$

or

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[d, "\vartheta\times\vartheta"'] & \Lambda \arrow[d, "\theta"] \arrow[dd, bend left, "\vartheta"] \\
\mathcal{F} \arrow[r, "\phi"] \arrow[d, "\mathrm{T}"'] & \Lambda \arrow[d, "\chi_{\frac{1}{2}}"] \\
\mathcal{F} \arrow[r, "\phi"] & \Lambda
\end{tikzcd} \tag{4.8}$$

We therefore get a map $\phi/\vartheta : \mathcal{F}/\psi \to \Lambda/\vartheta$, defined by $\phi/\vartheta \circ q = q \circ \phi$. The induced mapping on homology $\phi/\vartheta_* : H_i(\mathcal{F}/\psi; R) \to H_i(\Lambda/\vartheta; R)$ thus satisfies

$$\phi/\vartheta_* \circ q_* = q_* \circ \phi_* = q_* \circ \phi_{s*}$$

for any $s \in (0, 1)$.

5. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-action $\vartheta \times \theta$

On Λ we define a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action named $\vartheta \times \theta$ which is given by the involutions $\gamma \mapsto \bar{\gamma}$, $\gamma \mapsto \frac{1}{2}.\bar{\gamma}$, $\gamma \mapsto \frac{1}{2}.\gamma$. More precisely, we consider the group homomorphism $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to Aut(\Lambda)$ given by

$$\begin{aligned}
(0,0) &\mapsto id_\Lambda, \\
(1,0) &\mapsto \vartheta, \\
(0,1) &\mapsto \theta, \\
(1,1) &\mapsto \vartheta \circ \theta = \chi_{\frac{1}{2}}
\end{aligned}$$

for $\mathbb{Z}_2 = \{0, 1\}$. It is easily verified that this indeed is a group homomorphism. Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ has the discrete topology the above defines a continuous group action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on Λ.

For this action we content ourselves with considering the associated transfer product. That is, in Chapter 4 we will consider the product

$$P_{\vartheta\times\theta} : H_i\big(\Lambda/(\vartheta \times \theta); R\big) \times H_j\big(\Lambda/(\vartheta \times \theta); R\big) \to H_{i+j-n}\big(\Lambda/(\vartheta \times \theta); R\big). \tag{5.1}$$

6. Construction of the $\mathbb{Z}_2$-equivariant loop bracket B

Unlike for the other products, here we use equivariant homology and imitate the construction of the S^1-equivariant product described in [**GH09**, Section 17.4].
We consider an arbitrary $\mathbb{Z}_2$-action on $\Lambda = \Lambda M$ for some compact manifold M. $\mathbb{Z}_2$-equivariant homology is the singular homology of the so-called homotopy quotient $\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ of Λ. Here $E\mathbb{Z}_2 \to B\mathbb{Z}_2$ is the universal $\mathbb{Z}_2$-bundle and $\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ is the quotient of $\Lambda \times E\mathbb{Z}_2$ by the

diagonal action $\big(g,(\gamma,x)\big) \mapsto (g\gamma, gx)$. We can take $E\mathbb{Z}_2 = S^\infty$ and $B\mathbb{Z}_2 = \mathbb{R}P^\infty$ ([**Hus94**, Chapter 4, Example 11.3]).
The commutative diagram

$$\begin{tikzcd}
\Lambda \times E\mathbb{Z}_2 \arrow[r] \arrow[d, "p"] & E\mathbb{Z}_2 \arrow[d] \\
\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \arrow[r] & B\mathbb{Z}_2
\end{tikzcd}$$

is a pullback ([**tD08**, Proposition 14.1.3 or Proposition 14.1.10]) and hence $p : \Lambda \times E\mathbb{Z}_2 \to \Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ is a principal $\mathbb{Z}_2 = S^0$-bundle.

We can thus define the following product on equivariant homology $H_*^{\mathbb{Z}_2}(\Lambda) := H_*(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2)$: Consider a disk bundle $P : D \to \Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ which when restricted to its sphere bundle is $p : \Lambda \times E\mathbb{Z}_2 \to \Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2$. We can take D to be the mapping cylinder of p ([**Spa95**, Chapter 5, Section 7, Example 3]). Let

$$\Phi : H_i(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2) \xrightarrow{\cong} H_{i+1}(D, \Lambda \times E\mathbb{Z}_2)$$

be the Thom isomorphism of that disk bundle.

Definition 6.1. Let M be a connected, compact n-manifold and let $\Lambda = \Lambda M$ denote its free loop space. We define the loop bracket

$$B : H_i^{\mathbb{Z}_2}(\Lambda; \mathbb{Z}_2) \times H_j^{\mathbb{Z}_2}(\Lambda; \mathbb{Z}_2) \to H_{i+j-n}^{\mathbb{Z}_2}(\Lambda; \mathbb{Z}_2) \tag{6.1}$$

is defined by

$$\begin{tikzcd}
H_i^{\mathbb{Z}_2}(\Lambda;\mathbb{Z}_2) \times H_j^{\mathbb{Z}_2}(\Lambda;\mathbb{Z}_2) \arrow[d, "="'] \\
H_i(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2;\mathbb{Z}_2) \times H_j(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2;\mathbb{Z}_2) \arrow[d, "\Phi\times\Phi"'] \arrow[dd, bend left=60, "\partial\circ\Phi\times\partial\circ\Phi"] \\
H_{i+1}(D, \Lambda \times E\mathbb{Z}_2;\mathbb{Z}_2) \times H_{j+1}(D, \Lambda \times E\mathbb{Z}_2;\mathbb{Z}_2) \arrow[d, "\partial\times\partial"'] \\
H_i(\Lambda \times E\mathbb{Z}_2;\mathbb{Z}_2) \times H_j(\Lambda \times E\mathbb{Z}_2;\mathbb{Z}_2) \arrow[d, "\cong"'] \\
H_i(\Lambda;\mathbb{Z}_2) \times H_j(\Lambda;\mathbb{Z}_2) \arrow[d, "\text{Chas - Sullivan product}"'] \\
H_{i+j-n}(\Lambda;\mathbb{Z}_2) \arrow[d, "\cong"'] \\
H_{i+j-n}(\Lambda \times E\mathbb{Z}_2;\mathbb{Z}_2) \arrow[d, "p_*"'] \\
H_{i+j-n}(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2;\mathbb{Z}_2) \arrow[d, "="'] \\
H_{i+j-n}^{\mathbb{Z}_2}(\Lambda;\mathbb{Z}_2)
\end{tikzcd}$$

where we are using that $E\mathbb{Z}_2 = S^\infty$ is contractible ([**AGP02**, Chapter 11, Section 1] or more generally that EG is contractible for any group [**tD08**, Proposition 14.4.6]).

The map $\partial \circ \Phi$ appears in the Gysin sequence associated to sphere bundles ([**Spa95**, Chapter 5, Section 7, Theorem 11]). Here ∂ is the boundary homomorphism of the long exact homology sequence of the pair $(D, \Lambda \times E\mathbb{Z}_2)$.
Note that if $\Lambda = \Lambda M$ is simply-connected, so is $\Lambda \times E\mathbb{Z}_2$ as $\pi_1(\Lambda \times E\mathbb{Z}_2) \cong \pi_1(\Lambda) \times \pi_1(E\mathbb{Z}_2) = 0$ ([**Hat02**, Proposition 4.2]). This holds when M is 2-connected, so for example for S^n with $n > 2$ ([**Kli78**, Corollary 2.1.5]). It follows that $p : \Lambda \times E\mathbb{Z}_2 \to \Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ is a nonorientable sphere bundle ([**tD08**, Proposition 3.2.10]) and hence to define the above product we have to consider homology with coefficients in $\mathbb{Z}_2$ or even local coefficients. In particular, this product is not defined for rational coefficients.

It is interesting to note that this is actually a transfer product:
We can also look at the long exact transfer sequence of the 2-fold covering $\mathbb{Z}_2 \to \Lambda \times E\mathbb{Z}_2 \xrightarrow{p} \Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2$, that is the long exact sequence in homology

$$\ldots \longrightarrow H_{i+1}^{\mathbb{Z}_2}(\Lambda;\mathbb{Z}_2) \longrightarrow H_i^{\mathbb{Z}_2}(\Lambda;\mathbb{Z}_2) \xrightarrow{tr} H_i(\Lambda \times E\mathbb{Z}_2;\mathbb{Z}_2) \xrightarrow{p_*} H_i^{\mathbb{Z}_2}(\Lambda;\mathbb{Z}_2) \longrightarrow \ldots,$$

induced by the short exact sequence of chain groups

$$0 \longrightarrow C_i(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2; \mathbb{Z}_2) \xrightarrow{tr} C_i(\Lambda \times E\mathbb{Z}_2; \mathbb{Z}_2) \xrightarrow{p_\sharp} C_i(\Lambda \times_{\mathbb{Z}_2} E\mathbb{Z}_2; \mathbb{Z}_2) \longrightarrow 0.$$

Here tr is the transfer of an actual covering. It is a chain map ([**Hat02**, Section 3.G.]). The map it induces is also denoted by tr here. The transfer we introduced for ramified coverings generalizes this one in that both have the properties listed in the section on transfers. In particular with $\mathbb{Z}_2$ coefficients we have $ker(p_*) = im(tr)$ and the above sequence is exact. Comparing with the Gysin sequence used in the definition of the product B, we see that $tr = \partial \circ \Phi$ ([**Hat02**, Proof of Proposition 2B.6] or [**MS74**] Corollary 12.3). The relation of B to $P_{\mathbb{Z}_2}$ is unclear to the author.

CHAPTER 4

The equivariant homology of ΛS^n

In this chapter we do the following:

(1) In the first section we present some notions and theorems of equivariant Morse theory that we are going to use.
(2) In the second section we state a theorem about the transfer product P_θ on $H_*(\Lambda S^n/\theta;\mathbb{Q})$. The proof will be given in section three.
(3) In section three we compute $H_*(\Lambda S^n/\vartheta;\mathbb{Q})$. If M is a simply-connected, even-dimensional manifold and if we use rational coefficients, we have $A_\vartheta = P_\vartheta$. We show that there exists a nonnilpotent class $\eta \in H_*(\Lambda S^n/\vartheta;\mathbb{Q})$ for the product $A_\vartheta = P_\vartheta$ when n is even. The class η corresponds to Θ^2, where Θ is the nonnilpotent class of the Chas-Sullivan product on $H_*(\Lambda S^n)$. On odd spheres, the product P_ϑ also has a nonnilpotent class μ which comes form Θ.
(4) The rest of the chapter is devoted to other products, namely the product $P_{\vartheta\times\theta}$ and the bracket B.

1. Equivariant Morse theory

We have seen before that if for some metric g on M the energy $E = E_g$ is a Morse-Bott function, then the following is true ([**Kli78**, Theorem 2.4.10 and Corollary 2.4.11]):
Let $c \in [a,b]$ be the only critical value of E in $[a,b]$. Since E is Morse-Bott, $E^{-1}(c) = \bigsqcup_{i=1}^k B_i$, where each $B_i \subset \Lambda M$ is a nondegenerate connected submanifold. Then $\Lambda M^{\leq b}$ is homotopy equivalent to $\Lambda M^{\leq a} \cup_i D\Gamma_i^-$, where the latter is a disjoint attachment along the boundaries and $\Gamma_i^- \to B_i$ is the negative normal bundle of B_i. More precisely, let $i : \Lambda M^{\leq a} \cup_i D\Gamma_i \hookrightarrow \Lambda M^{\leq b}$ be the inclusion and $r : \Lambda M^{\leq b} \to \Lambda M^{\leq a} \cup_i D\Gamma_i$ be the retraction, then $r \circ i = id$ and $i \circ r \cong id$. If these homotopies are homotopies through equivariant maps, then homotopy equivalences on quotients and homotopy quotients are induced. We are going to see that this is actually the case.

First some generalities: Let $A \subset X$ be a subset of a topological space. Let $r : X \to A$ be a deformation retraction of X onto A, i.e. $r \circ i = id_A$ and $i \circ r \simeq id_X$, where $i : A \hookrightarrow X$ is the inclusion. Moreover, let X be a G-space for some topological group G, i.e. the action $G \times X \to X$ is continuous. Let $H : X \times [0,1] \to X$ be the homotopy $i \circ r \simeq id_X$, i.e. $H(x,0) = i \circ r(x)$, $H(x,1) = x$.
Assume now that H is an G-equivariant map where we consider the diagonal action on $X \times [0,1]$ and where $[0,1]$ carries the trivial G-action. Then the adjoint path $H^\sharp : [0,1] \to C^0(X,X)$ maps into the subspace of G-maps, because

$$gH_t(x) = gH^\sharp(t)(x) = gH(x,t) = H(gx,gt) = H(gx,t) = H^\sharp(t)(gx) = H_t(gx).$$

We write $i \circ r \simeq_G id_X$. H induces a continuous map $H/G : (X \times [0,1])/G \to X/G$. We have that $(X \times [0,1])/G = X \times_G [0,1] \cong X/G \times [0,1]$. This holds since

- Let $p : X \to X/G$, then $q : X \times [0,1] \xrightarrow{p \times id} X/G \times [0,1]$ is a quotient map, because $[0,1]$ is (locally) compact. Note that q is an open map.
- Let $Q : X \times [0,1] \to X \times_G [0,1]$ be the quotient map, which is again open, being the quotient by a continuous group action. Then the map $\tilde{q} : X \times_G [0,1] \to X/G \times [0,1]$, $[(x,t)] \mapsto ([x],t)$ induced by q is bijective, continuous and open, hence a homeomorphism. Its inverse is $\tilde{Q}$, the map induced by Q:

$$\begin{tikzcd} & X \times [0,1] \arrow[dl, "Q"'] \arrow[dr, "q"] & \\ X \times_G [0,1] \arrow[rr, bend left, "\tilde{q}"] & & X/G \times [0,1]. \arrow[ll, bend left, "\tilde{Q}"] \end{tikzcd}$$

We consider $F := H/G \circ \tilde{Q} : X/G \times [0,1] \to X/G$ and its adjoint map $F^\sharp : [0,1] \to C^0(X/G, X/G)$ and have

$$\begin{aligned} F_t([x]) &= F^\sharp(t)([x]) = F([x],t) = H/G\big([(x,t)]\big) = (p \circ H)(x,t), \\ F_0([x]) &= (p \circ H)(x,0) = (p \circ i \circ r)(x) = [(i \circ r)(x)] = \big((i \circ r)/G\big)([x]), \\ F_1([x]) &= (p \circ H)(x,1) = (p \circ id_X)(x) = [x] = id_{X/G}\big([x]\big). \end{aligned}$$

So F is a homotopy between $(i \circ r)/G$ and $id_{X/G}$.
We further assume that i and r are both G-equivariant. This implies that $i \circ r$ is G-equivariant. Also i G-equivariant, i.e. $gi(a) = i(ga)$ for all $g \in G$, is clearly equivalent to A being a G-invariant subspace, i.e. itself a G-space. By the uniqueness of the map $(i \circ r)/G$, we thus get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "r"'] \arrow[d] \arrow[rr, bend left, "i \circ r"] & A \arrow[r, "i"'] \arrow[d] & X \arrow[d] \\ X/G \arrow[r, "r/G"] \arrow[rr, bend right, "(i \circ r)/G"'] & A/G \arrow[r, "i/G"] & X/G. \end{tikzcd}$$

So, in this case, F is a homotopy between $i/G \circ r/G$ and $id_{X/G}$. On the other hand $r \circ i = id_A$ implies that $r/G \circ i/G = id_{A/G}$. That is i/G and r/G are homotopy inverse to one another and hence induce isomorphisms on homology which are inverse to one another:

$$\begin{tikzcd} H_i(X/G) \arrow[r, bend left, "(r/G)_*", "\cong"'] & H_i(A/G). \arrow[l, bend left, "(i/G)_*", "\cong"'] \end{tikzcd}$$

Let now $p_G : EG \to BG$ be the universal G-bundle. It classifies principal G-bundles and its existence is assured by e.g. Section 12 of Chapter 4 of [**Hus94**]. If $f : X \to Y$ is a G-equivariant map between G-spaces, then so is $f \times id : X \times EG \to Y \times EG$, where we consider $X \times EG$ as a G-space via the diagonal action. (The diagonal action is $G \times X \times EG \xrightarrow{\Delta \times id \times id} G \times G \times X \times EG \cong G \times X \times G \times EG \to X \times EG$ and hence continuous.) The base space

$X_G := X \times_G EG$ of the orbit map $X \times EG \to (X \times EG)/G =: X \times_G EG$ is called the homotopy quotient of X. The homology of the homotopy quotient is known as the equivariant homology of the space X:

$$H_i^G(X) := H_i(X \times_G EG).$$

If two G-space X, Y are G-homotopy equivalent, then so are $X \times EG$ and $Y \times EG$ and we have

$$H_i(X/G) \cong H_i(Y/G),$$
$$H_i^G(X) \cong H_i^G(Y).$$

The main reason for considering $X \times EG$ instead of X is that G acts freely on $X \times EG$, no matter how it acts on X.

The relevant theorems. Let (M, g) be a compact Riemannian manifold and let $G \subset O(2)$ be any subgroup acting in the obvious way on ΛM. According to [**Hin84**] we can do equivariant Morse theory with E_g on $(\Lambda M, g_1)$ since we have:

- ΛM is a G-manifold: ΛM is paracompact, Hausdorff manifold and the action of G is continuous and each $g \in G$ acts differentiably ([**Kli78**, Lemma 2.2.1 and Theorem 2.2.5]).
- ΛM is a complete G-Riemannian manifold: G acts by isometries on $(\Lambda M, g_1)$ and $(\Lambda M, g_1)$ is complete ([**Kli78**, Theorem 1.4.5, Lemma 2.2.1 and Theorem 2.2.5]).
- $E = E_g$ is differentiable and invariant under G ([**Kli78**, Lemma 1.3.9, Lemma 2.2.1 and Theorem 2.2.5]).
- E satisfies Condition C_G: C_G is like the usual condition C (i.e. the Palais-Smale condition introduced in Chapter 1) but now the image of the subsequence $\{p_{n_k}\}$ in $\Lambda M/G$ is required to converge. Here C_G holds since C holds and C implies C_G (see [**Kli78**, Theorem 1.4.7]).

By Theorems A and B in Section 2 of [**Hin84**] we then have

- If the interval $[a, b]$ contains no critical values of E then
$$H_i(\Lambda M^{\leq b}/G, \Lambda M^{\leq a}/G) = 0,$$
$$H_i^G(\Lambda M^{\leq b}, \Lambda M^{\leq a}) = 0.$$
- Let a, b be regular values of E. If the interval $[a, b]$ contains exactly one critical value $c \in (a, b)$ in the interior and if $Cr(E) \cap E^{-1}(c) = \bigsqcup_r B_r$ where B_r are nondegenerate critical G-submanifolds then
$$H_i(\Lambda M^{\leq b}/G, \Lambda M^{\leq a}/G) \cong \bigoplus_r H_i(\Gamma_r^-/G, (\Gamma_r^- - B_r)/G),$$
$$H_i^G(\Lambda M^{\leq b}, \Lambda M^{\leq a}) \cong \bigoplus_r H_i^G(\Gamma_r^-, \Gamma_r^- - B_r).$$
Here $\Gamma_r^- \to B_r$ is the negative normal bundle over B_r.

This is a consequence of the following facts ([**Kli78**, Section 2.4] and [**Hin84**]):

- Let B be a nondegenerate critical submanifold. The normal bundle $\Gamma \to B$ is a G-vector bundle: This holds since the submanifold B is itself a G-manifold and hence equivariantly embedded.

- Since B is nondegenerate, there exists a G-orthogonal splitting $\Gamma \cong \Gamma^+ \oplus \Gamma^-$: This means that the splitting is orthogonal with respect to the metric g_1 and that it is G-invariant. Nondegenerate means nondegenerate in normal direction, i.e. $TB = \Gamma^0$, where Γ^0 denotes the kernel of the Hessian of E.
- Since the energy E satisfies the Palais-Smale condition, assuming that B is connected implies that E is constant on B, say $E(B) = c$. The Morse Lemma implies that $\Lambda M^{\leq c}$ is G-diffeomorphic to $\Lambda M^{\leq c-\varepsilon} \cup (D\Gamma^+ \oplus D\Gamma^-)$. Here $\Lambda M^{\leq c-\varepsilon} \cup (D\Gamma^+ \oplus D\Gamma^-)$ denotes $\Lambda M^{\leq c-\varepsilon}$ with a handle bundle of type $D\Gamma^+ \oplus D\Gamma^-$ equivariantly attached as explained in Section 2.4 of [**Kli78**]. $D\Gamma^+$ and $D\Gamma^-$ denote the closed unit disk bundles. Moreover, $\Lambda M^{\leq c-\varepsilon} \cup (D\Gamma^+ \oplus D\Gamma^-)$ is G-homotopy equivalent to $\Lambda M^{\leq c-\varepsilon} \cup_{\partial D\Gamma^-} D\Gamma^-$.
- We then have

$$
\begin{tikzcd}
(\Gamma^-, \Gamma^- - B) \\
(D\Gamma^-, \partial D\Gamma^-) \arrow[u, hook] \arrow[d, hook] \\
(\Lambda M^{\leq c-\varepsilon} \cup D\Gamma^-, \Lambda M^{\leq c-\varepsilon}) \arrow[d, hook] \\
(\Lambda M^{\leq c-\varepsilon} \cup (D\Gamma^+ \oplus D\Gamma^-), \Lambda M^{\leq c-\varepsilon}) \arrow[d, "\cong"'] \\
(\Lambda M^{\leq c}, \Lambda M^{\leq c-\varepsilon})
\end{tikzcd}
$$

  and since G acts by isometries of g_1, all of the above maps are G-equivariant and have G-equivariant inverses, at least homotopical. Thus the homology groups of the G-quotients and G-homotopy quotients of the above pairs are isomorphic.

Hence, if there is only one connected critical nondegenerate submanifold B between the levels a and b, then going form a to b the "topology changes" by

$$H_i(\Gamma^-/G, (\Gamma^- - B)/G)$$

or

$$H_i^G(\Gamma^-, \Gamma^- - B) \cong H^G_{i-rank(\Gamma^-)}(B),$$

respectively. The last isomorphism is the Thom isomorphism which of course only holds when the bundle is orientable. This is due to the fact that

$$\Gamma^- \times_G EG \to B \times_G EG$$

still is a vector bundle since G acts freely on $B \times EG$ and $B \times EG \to B \times_G EG$ is a principal G-bundle ([**tD87**, Chapter I, Proposition 9.4]). On the other hand,

$$\Gamma/G \to B/G$$

is in general not a vector bundle anymore, but if the action on B is free and $B \to B/G$ is a principal G-bundle then it is. This holds for example when G is finite and discrete and B is Hausdorff ([**LS15**, Folgerung 9.5]).

2. Orientation reversal and shift on spheres: Computation of $\big(H_*(\Lambda S^n/\theta;\mathbb{Q}), P_\theta\big)$

Let S^n be endowed with the standard Riemannian metric.

We consider the $\mathbb{Z}_2$-action

$$\theta : \Lambda \to \Lambda,\ \theta(\gamma) = \frac{1}{2}.\bar{\gamma} = (\chi_{\frac{1}{2}} \circ \vartheta)(\gamma).$$

Since $\chi_{\frac{1}{2}} \circ \vartheta = \vartheta \circ \chi_{\frac{1}{2}}$ we have

$$\theta(\gamma) = \frac{1}{2}.\bar{\gamma} = \overline{\frac{1}{2}.\gamma}.$$

Since this is a $\mathbb{Z}_2$-action, it is free away from the fixed point set

$$\mathrm{Fix}(\theta) := \{\gamma \in \Lambda |\, \gamma(t) = \gamma(1-t-\frac{1}{2})\}.$$

There are no closed geodesics in $\mathrm{Fix}(\theta)$: Let $(\chi_{\frac{1}{2}} \circ \vartheta)(\gamma) = \theta(\gamma) = \gamma$. This is equivalent to $\vartheta\big((\chi_{\frac{1}{4}})^{-1}(\gamma)\big) = (\chi_{\frac{1}{4}})^{-1}(\gamma)$ as $\chi_t \circ \vartheta = \vartheta \circ (\chi_t)^{-1}$. Thus

$$\gamma \in \mathrm{Fix}(\theta) \Longleftrightarrow (\chi_{\frac{1}{4}})^{-1}(\gamma) \in \mathrm{Fix}(\vartheta).$$

As the energy E_g is invariant under $\chi_{\frac{1}{4}}$, γ is a closed geodesic if and only if $(\chi_{\frac{1}{4}})^{-1}(\gamma)$ is one. As $\mathrm{Fix}(\vartheta) := \{\gamma \in \Lambda |\, \gamma(t) = \gamma(1-t)\, \forall t \in [0,1]\} = \{\gamma \in \Lambda |\, \gamma(\frac{1}{2}+t) = \gamma(\frac{1}{2}-t)\, \forall t \in [0,1]\}$ it cannot contain any closed geodesic as such curves are not smooth ([**Rad92**, page 12]). Therefore also $\mathrm{Fix}(\theta)$ does not contain closed geodesics.
In particular, the restrictions of the $\mathbb{Z}_2$-actions θ and ϑ to the nondegenerate critical submanifold of r-fold covered great circles B_r are free for any r. Let $\gamma_r = c_{p,rv}$ be the r-fold covered great circle corresponding to $(p,v) \in V_2(\mathbb{R}^{n+1}) \cong T^1S^n \cong B_r$. There are two cases, depending on whether r is even or odd, namely

$$\theta(\gamma_r) = \begin{cases} \overline{\gamma_r} = \vartheta(\gamma_r), & \text{if } r \text{ is even} \\ \overline{\frac{1}{2}.\gamma_r} = \vartheta \circ \chi_{\frac{1}{2}}(\gamma_r), & \text{if } r \text{ is odd} \end{cases}.$$

This holds since if r is even, i.e. $\frac{r}{2} \in \mathbb{N}$, then B_r is pointwise fixed under $\chi_{\frac{1}{2}}$: Let $\gamma := \gamma_1$, then

$$\chi_{\frac{1}{2}}(\gamma_r)(t) = \gamma_r(t+\tfrac{1}{2}) = \gamma\big(r(t+\tfrac{1}{2})\big) = \gamma(rt+\tfrac{r}{2}) = \gamma(rt) = \gamma_r(t).$$

So on B_r for r even, the two actions coincide. Thus, under the diffeomorphism $f_r : V_2(\mathbb{R}^{n+1}) \to B_r$, the involution θ corresponds to

$$\theta(p,v) = \begin{cases} (p,-v) = \vartheta(p,v), & \text{if } r \text{ is even} \\ (-p,v), & \text{if } r \text{ is odd} \end{cases}$$

for $(p,v) \in V_2(\mathbb{R}^{n+1})$.

This close relation between the two actions θ and ϑ together with

$$\vartheta_* = \theta_* : H_*(\Lambda) \to H_*(\Lambda)$$

and

$$\big(H_*(\Lambda/\vartheta;\mathbb{Q}), P_\vartheta\big) \cong \big(H_*(\Lambda/\theta;\mathbb{Q}), P_\theta\big)$$

(see Theorem 2.7, Section 2 of Chapter 3) lead to

THEOREM 2.1. *Let $n > 2$ and let θ be the orientation reversal plus the shift of the starting point by $1/2$ of loops on ΛS^n. Then*

- *for n odd, there exists a generator ν of $H_{3n-2}(\Lambda S^n/\theta;\mathbb{Q})$ which is not nilpotent in the algebra $(H_*(\Lambda/\theta;\mathbb{Q}), P_\theta)$. More precisely, for every $k \in \mathbb{N}$, ν^k is a generator of $H_{2k(n-1)+n}(\Lambda/\theta;\mathbb{Q})$.*
  Moreover, multiplication with ν, i.e.
  $$P_\theta(\cdot,\nu) : H_i(\Lambda S^n/\theta;\mathbb{Q}) \to H_{i+2n-2}(\Lambda S^n/\theta;\mathbb{Q})$$
  is an isomorphism for $i \geq 0$.
- *for n even, there exists a generator η of $H_{5n-4}(\Lambda S^n/\vartheta;\mathbb{Q})$ which is not nilpotent in the algebra $(H_*(\Lambda S^n/\vartheta;\mathbb{Q}), P_\theta)$. More precisely, for every $k \in \mathbb{N}$, η^k is a generator of $H_{(4k+1)n-4k}(\Lambda S^n/\theta;\mathbb{Q})$.*
  Moreover, multiplication with η, i.e.
  $$P_\theta(\cdot,\eta) : H_i(\Lambda S^n/\theta;\mathbb{Q}) \to H_{i+4n-4}(\Lambda S^n/\theta;\mathbb{Q})$$
  is an isomorphism for $i > 0$.

PROOF. This theorem will be a corollary of an analogous theorem (Theorem 3.18 below) with θ replaced by ϑ presented in the next section. ☐

3. Orientation reversal on spheres: Computation of $\big(H_*(\Lambda S^n/\vartheta;\mathbb{Q}), P_\vartheta\big)$

Let S^n be endowed with the standard Riemannian metric.

The critical values of $E_{g_{st}}$ are given by $2\pi^2 r^2$ for $r \in \mathbb{N}_0$ and we have

$$H_i(\Lambda S^{n\leq 2\pi^2r^2}/\vartheta, \Lambda S^{n<2\pi^2r^2}/\vartheta) \cong H_i(D\Gamma_r^-/\vartheta, \partial D\Gamma_r^-/\vartheta), \tag{3.1}$$

$$H_i^{\mathbb{Z}_2}(\Lambda S^{n\leq 2\pi^2r^2}, \Lambda S^{n<2\pi^2r^2}) \cong H_i^{\mathbb{Z}_2}(D\Gamma_r^-, \partial D\Gamma_r^-). \tag{3.2}$$

for $r \geq 1$. Recall that this holds since $\mathbb{Z}_2 \subset O(2)$ acts isometrically. Also keep in mind that here in this section

$$H_i^{\mathbb{Z}_2}(\Lambda S^{n\leq 2\pi^2r^2}, \Lambda S^{n<2\pi^2r^2}) = H_i(\Lambda S^{n\leq 2\pi^2r^2} \times_{\mathbb{Z}_2} E\mathbb{Z}_2, \Lambda S^{n<2\pi^2r^2} \times_{\mathbb{Z}_2} E\mathbb{Z}_2)$$

where the diagonal action on $\Lambda S^n \times E\mathbb{Z}_2 = \Lambda S^n \times S^\infty$ is generated by the involution $\vartheta\times$(antipodal map).

If the vector bundle $\Gamma_r^- \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \to B_r \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ is orientable we have

$$H_i^{\mathbb{Z}_2}(D\Gamma_r^-, \partial D\Gamma_r^-;\mathbb{Z}) \cong H_{i-\lambda_r}^{\mathbb{Z}_2}(B_r;\mathbb{Z}),$$

if not we only have

$$H_i^{\mathbb{Z}_2}(D\Gamma_r^-, \partial D\Gamma_r^-;\mathbb{Z}_2) \cong H_{i-\lambda_r}^{\mathbb{Z}_2}(B_r;\mathbb{Z}_2).$$

Here B_r is the non-degenerate critical manifold consisting of the r-fold covered great circles and $\lambda_r = (2r-1)(n-1)$ is the index of an r-fold covered great circle. The space of r-fold covered great circles $B_r \subset \Lambda S^n$ is diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^{n+1})$, which in turn is diffeomorphic to the unit tangent bundle T^1S^n of S^n (see Section 3 of Chapter 1).

3.1. Orientability of the quotient negative normal bundles. Let $p_r : \Gamma_r^- \to B_r$ be the negative normal bundle; its fibres have dimension λ_r and $\mathbb{Z}_2$ acts continuously (as $\mathbb{Z}_2$ is discrete) on it by isometric bundle isomorphisms via the differential of the isometric involution ϑ. We also consider the $\mathbb{Z}_2$-vector bundle $p_r \times id : \Gamma_r^- \times E\mathbb{Z}_2 \to B_r \times E\mathbb{Z}_2$.
We denote the involution $(p, v) \mapsto (p, -v)$ on T^1S^n by ϑ as well. The associated $\mathbb{Z}_2$-action is free and the map $f_r : T^1S^n \to B_r, (p, v) \mapsto c_{p,rv}$ is a $\mathbb{Z}_2$-equivariant map between free $\mathbb{Z}_2$-spaces. Hence, for both bundles $\Gamma_r^- \to B_r$ and $\Gamma_r^- \times E\mathbb{Z}_2 \to B_r \times E\mathbb{Z}_2$ the action on the base is free: it happens to be free on B_r in this specific case of orientation reversal of loops and it is free on $X \times EG$ for any group G and any G-space X. Thus, for both bundles the quotients will be ordinary vector bundles and they fit into the following pullback diagrams ([**Kna13**, Proposition 2.8.6.], or, for a more general setting, see [**tD87**, Chapter I, Proposition 9.4]):

$$\begin{array}{ccc} \Gamma_r^- \times E\mathbb{Z}_2 & \longrightarrow & \Gamma_r^- \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \\ \downarrow & & \downarrow \\ B_r \times E\mathbb{Z}_2 & \longrightarrow & B_r \times_{\mathbb{Z}_2} E\mathbb{Z}_2, \end{array} \qquad \begin{array}{ccc} \Gamma_r^- & \longrightarrow & \Gamma_r^-/\mathbb{Z}_2 = \Gamma_r^-/\vartheta \\ \downarrow & & \downarrow \\ B_r & \longrightarrow & B_r/\mathbb{Z}_2 = B_r/\vartheta. \end{array}$$

Here the horizontal maps are the quotient maps, which are in fact 2-sheeted covering maps and $\mathbb{Z}_2$-principal bundles ([**LS15**, Folgerung 9.5]). As the left-hand bundle projections in both of the diagrams are $\mathbb{Z}_2$-equivariant maps it follows directly from Proposition 14.1.10 in [**tD08**] that the above diagrams are pullback squares.
It follows immediately that orientability of $\Gamma_r^- \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \to B_r \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ implies orientability of $\Gamma_r^- \times E\mathbb{Z}_2 \to B_r \times E\mathbb{Z}_2$ and orientability of $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ implies orientability of $p_r : \Gamma_r^- \to B_r$. If the action of $\mathbb{Z}_2$ preserves orientations, the orientations can also be "pushed foreward":

Lemma 3.1. *Let G be a finite, discrete topological group acting continuously and freely on a topological space B. Let B be Hausdorff. Let $p : V \to B$ be a real G-vector bundle of finite rank λ, that is, $p : V \to B$ is a real vector bundle with fibre $\mathbb{R}^\lambda$, V is also a G-space, p is G-equivariant and G acts linearly on fibres.*
Then $p/G : V/G \to B/G$ is a vector bundle and it is orientable if and only if $V \to B$ is orientable and G preserves orientation.

Note that this applies to the bundles $\Gamma_r^-/G \to B_r/G$ for any finite subgroup G of $O(2)$ acting freely on B_r and to the bundles $\Gamma_r^- \times_G EG \to B_r \times_G EG$ for any finite subgroup G of $O(2)$.

Proof. The finite compact Lie group G acts freely and properly on the Hausdorff space B, hence $G \to B \to B/G$ is a principal G-bundle (see e.g. [**LS15**, Folgerung 9.5]). Let $U \subset B/G$ be an open subset above which the principal G-bundle $q : B \to B/G$ is trivial, i.e. $q^{-1}(U) = \bigsqcup_{g \in G} U_g \cong_G U \times G$ ($\cong_G$ stands for G-equivariantly homeomorphic). Here each $U_g \subset q^{-1}(U) \subset B$ corresponds to $U \times \{g\}$ under the trivialization and is mapped homeomorphically onto U by the quotient map q. G permutes the sheets U_g. (If B is path-connected, as in our case, G is the automorphism group of the covering q (see e.g. [**Hat02**, Proposition 1.40]). Let $e \in G$ denote the unit element and let U be small enough such that $p : V \to B$ is trivial above U_e. Then also $V_{|U_g} = p_{|U_g} : p^{-1}(U_g) \to U_g$ is trivial for any $g \in G$

since they are all isomorphic via translation. A trivialization for $V_{|U_g}$ is for example

$$\begin{tikzcd}
p^{-1}(U_g) \arrow[r, "g^{-1}", "\cong"'] \arrow[d, "p"] & p^{-1}(U_e) \arrow[r, "h^{-1}", "\cong"'] \arrow[d, "p"] & U_e \times \mathbb{R}^\lambda \arrow[r, "g\times id", "\cong"'] \arrow[d, "pr_{U_e}"] & U_g \times \mathbb{R}^\lambda \arrow[d, "pr_{U_g}"] \\
U_g \arrow[r, "\cong", "g^{-1}"'] & U_e \arrow[r, "="] & U_e \arrow[r, "\cong", "g"'] & U_g
\end{tikzcd}$$

where h is the bundle chart over U_e. We have

$$p^{-1}\big(q^{-1}(U)\big) = p^{-1}\big(\bigsqcup_{g\in G} U_g\big) = \bigsqcup_{g\in G} p^{-1}(U_g) = \bigsqcup_{g\in G} V_{|U_g}.$$

In the quotient V/G, these disjoint $V_{|U_g}$ are all identified:

$$\begin{tikzcd}
\bigsqcup_{g\in G} V_{|U_g} \arrow[r, "\tilde{q}"] \arrow[d, "p"'] & \tilde{q}(V_{|U_e}) \arrow[d, "p/G"] \\
\bigsqcup_{g\in G} U_g \arrow[r, "q"'] & U
\end{tikzcd}$$

commutes. Here $\tilde{q} : V \to V/G$ is the identification map (which is also a principal G-bundle). This identification is a bundle map since $\tilde{q} : V \to V/G$ is fibrewise bijective: For a point $b \in B$ the fibre over $[b] \in B/G$ is the vector space

$$(p/G)^{-1}([b]) = \tilde{q}\big(\bigsqcup_{g\in G} V_{gb}\big) = \{[v]|\; v \in V_b\} \cong V_b$$

where V_{gb} is the fibre of p over gb. We have a homeomorphism f defined by the composition

$$U \times \mathbb{R}^\lambda \xleftarrow[\cong]{q\times id} U_e \times \mathbb{R}^\lambda \xrightarrow[\cong]{h} V_{|U_e} \xrightarrow[\cong]{\tilde{q}} \tilde{q}(V_{|U_e}) = (p/G)^{-1}(U) = V/G_{|U}.$$

Here $\tilde{q}_{|V_{|U_e}} : V_{|U_e} \to V/G_{|U}$ is a homeomorphism as it is a continuous, bijective and open map, $V_{|U_e}$ being open in V. (f, U) is a bundle chart for V/G, that is

$$\begin{tikzcd}
V/G_{|U} \arrow[dr, "p/G"'] & & U \times \mathbb{R}^\lambda \arrow[ll, "f"'] \arrow[dl, "pr_U"] \\
& U &
\end{tikzcd}$$

commutes. This holds true since $(p/G \circ f) = (p/G) \circ \tilde{q} \circ h \circ (q \times id)^{-1} = q \circ p \circ h \circ (q \times id)^{-1} = q \circ pr_{U_e} \circ (q \times id)^{-1} = pr_U$. The above diagram then reads

$$\begin{tikzcd}
U \times G \times \mathbb{R}^{\lambda_r} \cong \bigsqcup_{g\in G}(U_g \times \mathbb{R}^{\lambda_r}) \arrow[dr] \arrow[ddr] & \bigsqcup_{g\in G} V_{|U_g} \arrow[l] \arrow[r, "\tilde{q}"] \arrow[d, "p"'] & V/G_{|U} \arrow[r] \arrow[d, "p/G"] & U \times \mathbb{R}^{\lambda_r} \arrow[dl] \\
& \bigsqcup_{g\in G} U_g \arrow[r, "q"'] \arrow[d] & U & \\
& U \times G \arrow[ur] & &
\end{tikzcd}$$

.

The fact that $\tilde{q}: V \to V/G$ is fibrewise bijective also implies that the commutative diagram

$$\begin{tikzcd} V \arrow[r, "\tilde{q}"] \arrow[d, "p"'] & V/G \arrow[d, "p/G"] \\ B \arrow[r, "q"'] & B/G \end{tikzcd}$$

is a pullback diagram (this also follows from [**tD08**, Propositon 14.1.10]). Hence $p/G : V/G \to B/G$ is a vector bundle.

We now address the problem of "pushing orientation classes down". That is, given a Thom class τ for the bundle $V \to B$, how to induce a Thom class of $V/G \to B/G$, if possible: Let $s : U \to U \times G \cong_G q^{-1}(U) = B_{|U}$ be a section of $B_{|U} \to U$ and $q^*s : V/G_{|U} \to V_{|U\times G} = V_{|B_{|U}}$ the pulled back section (see e.g. [**Hus94**, Chapter 2, Proposition 5.10]):

$$\begin{tikzcd} (V_U, V_U - B_U) \arrow[r, "\tilde{q}"'] \arrow[d, "p"'] & (V/G_U, V/G_U - U) \arrow[l, bend right, "q^*s"'] \arrow[d, "p/G"] \\ B_U \arrow[r, "q"'] & U \arrow[l, bend right, "s"'] \end{tikzcd}$$

where we use the simplifying notation B_U for $B_{|U}$ and V_U to denote $V_{|B_{|U}}$ and V/G_U for $V/G_{|U}$.

We now assume that $V \to B$ is oriented, that is it has a Thom class $\tau \in H^\lambda(V, V - B; \mathbb{Z})$. This Thom class then induces local orientations $(q^*s)^*(\tau_U) \in H^\lambda(V/G_U, V/G_U - U; \mathbb{Z})$, where τ_U stands for $\tau_{|V_U} \in H^\lambda(V_U, V_U - B_U; \mathbb{Z})$. To see this, first note that $\tilde{q} \circ q^*s = id_{V/G_U}$ implies that $(q^*s)^*$ is surjective and hence maps generators to generators. It also implies that $\tilde{q}^*$ is injective and that $\tilde{q}^*\big((q^*s)^*(\tau_U)\big) = \tau_U$ since $(q^*s)^*(\tau_U)$ and τ_U are generators. Let now $j_b : V_b \hookrightarrow V_U$ and $j_{[b]} : V/G_{[b]} \hookrightarrow V/G_U$, respectively, denote the inclusion of a fibre, then the diagram

$$\begin{tikzcd} H^\lambda(V_U, V_U - B_U) \arrow[d, "j_b^*"'] & H^\lambda(V/G_U, V/G_U - U) \arrow[l, "\tilde{q}^*"'] \arrow[d, "j_{[b]}^*"] \\ H^\lambda(V_b, V_b - 0) & H^\lambda(V/G_{[b]}, V/G_{[b]} - 0) \arrow[l, "\cong", "\tilde{q}^*"'] \end{tikzcd}$$

commutes. The lower horizontal map is an isomorphism since $\tilde{q}$ is a homeomorphism on fibres. This implies that $(q^*s)^*(\tau_U)$ restricts to generators on fibres and hence is a Thom class for $V/G_U \to U$. Let U be connected, then, as G is finite (and discrete), there are exactly $|G|$ possible sections, namely $U \to U \times G$, $[b] \mapsto ([b], g)$, one for each $g \in G$ (apply e.g. Theorem 8.1 in Chapter 4 of [**Hus94**] to the case of associating $B_{|U} \to U$ to itself). Now, if G preserves orientation, that is $g^*(\tau) = \tau$ for all $g \in G$, then $(q^*s)^*(\tau) = (q^*s)^*\big(g^*(\tau)\big) = (q^*g \cdot s)^*(\tau)$, where $g \cdot s : U \to U \times G$ is one of the other possible sections. The class $(q^*s)^*(\tau_U)$ is hence independent of choices and thus well-defined. Let us denote it by τ/G_U.

We must check that the local classes so defined associated to a trivializing cover of B/G agree on the overlaps. If so, they patch together to yield a global Thom class $\tau/G \in H^\lambda(V/G, V/G-$

B/G) as assured by e.g. Corollary 17 of Section 7 of Chapter 5 in [**Spa95**]. So let U and U' be two arbitrary elements of a trivializing cover. Then, the commutativity of the diagram

$$\begin{tikzcd}
(V_U, V_U - B_U) \arrow[d, "\tilde{q}"'] & (V_{U\cap U'}, V_{U\cap U'} - B_{U\cap U'}) \arrow[l, hook', "i_U"'] \arrow[r, hook, "i_{U'}"] \arrow[d, "\tilde{q}"'] & (V_{U'}, V_{U'} - B_{U'}) \arrow[d, "\tilde{q}"'] \\
(V/G_U, V_U/G - U) & (V/G_{U\cap U'}, V/G_{U\cap U'} - U\cap U') \arrow[l, hook', "i_U/G"'] \arrow[r, hook, "i_{U'}/G"] & (V/G_{U'}, V/G_{U'} - U')
\end{tikzcd}$$

implies that $(i_U/G)^*(\tau/G_U) = (i_{U'}/G)^*(\tau/G_{U'})$:

$$\begin{aligned}
\tilde{q}^*\big((i_U/G)^*(\tau/G_U)\big) &= i_U^*\big(\tilde{q}^*(\tau/G_U)\big) = i_U^*(\tau_U) = i_{U'}^*(\tau_{U'}) = i_{U'}^*\big(\tilde{q}^*(\tau/G_{U'})\big) \\
&= \tilde{q}^*\big((i_{U'}/G)^*(\tau/G_{U'})\big)
\end{aligned}$$

and since $\tilde{q}^*$ is injective we have $(i_U/G^) * (\tau/G_U) = (i_{U'}/G)^*(\tau/G_{U'})$. $\square$

Corollary 3.2. *The vector bundle $(p\times id)/G : V\times_G EG \to B\times_G EG$ is orientable if and only if $V \to B$ is orientable and G preserves orientation. Hence, if $p/G : V/G \to B/G$ is a vector bundle, then $p/G : V/G \to B/G$ is orientable if and only if $(p\times id)/G : V\times_G EG \to B\times_G EG$ is orientable.*

Proof. Let $\tau \in H^\lambda(V, V-B;\mathbb{Z})$ be a Thom class of the bundle $V\to B$. Let $1 \in H^0(EG)$ be a generator. An element $g\in G$ then acts on the Thom class

$$\tau\times 1 \in H^\lambda(V\times EG, (V-B)\times EG;\mathbb{Z}) \cong H^\lambda(V, V-B;\mathbb{Z})\otimes H^0(EG;\mathbb{Z})$$

via

$$(g\times g)^*(\tau\times 1) = g^*(\tau)\times g^*(1) = g^*(\tau)\times 1$$

as EG is path-connected. $\square$

Remark 3.3. The action of the compact Lie group $S^1 \subset O(2)$ on the smooth manifold B_r is unfortunately not free if $r>1$ and so $B_r \to B_r/S^1$ is a principal S^1-bundle if and only if $r=1$. For a fixed r the isotropy group of the S^1 action is $\mathbb{Z}_r$ for any element of B_r. Hence $S^1/\mathbb{Z}_r$ acts freely on B_r and $B_r \to B_r/(S^1/\mathbb{Z}_r) \cong B_r/S^1$ ([**Bre72**, Exercises for Chapter 1]) is a principal $S^1/\mathbb{Z}_r$-bundle. Hence $\Gamma_r^-/(S^1/\mathbb{Z}_r) \to B_r/(S^1/\mathbb{Z}_r) \cong B_r/S^1$ is an orientable vector bundle since the action of any element of S^1 is homotopic to the action of the identity as S^1 is path-connected. However, the relation of that bundle to the level homology is unclear to the author since $\mathbb{Z}_r$ might not be the isotropy group of Γ_r^-.

Since $\mathbb{Z}_2 \to B_r \to B_r/\vartheta$ is a principal $\mathbb{Z}_2$-bundle, the diagram

$$\begin{tikzcd}
B_r\times E\mathbb{Z}_2 \arrow[r] \arrow[d] & B_r \arrow[d] \\
B_r\times_{\mathbb{Z}_2} E\mathbb{Z}_2 \arrow[r, "\pi"] & B_r/\vartheta
\end{tikzcd}$$

is a pullback ([**tD08**, Proposition 14.1.3]) and $B_r\times_{\mathbb{Z}_2} E\mathbb{Z}_2 \xrightarrow{\pi} B_r/\vartheta$ is a fibre bundle ([**Hus94**, Chapter 4, Proposition 5.3]) with fibre $E\mathbb{Z}_2 \cong S^\infty$. Since the fibre is contractible (cite[Theorem 11.1.3]MR1908260) we have that π induces an isomorphism on all homotopy groups ([**Hat02**, Theorem 4.41]) and hence π is a weak homotopy equivalence. This shows that base and total space have isomorphic homology groups ([**Hat02**, Proposition 4.21]). π is even a homotopy equivalence: Since the base B_r/ϑ is a compact manifold, it can be given the structure of a CW-complex. Also $B_r\times_{\mathbb{Z}_2} E\mathbb{Z}_2$ has at least the homotopy type of CW-complex ([**FP90**, Theorem 5.4.2]) and hence the Whitehead theorem ([**Hat02**, Theorem 4.5]) assures that π is a homotopy equivalence. In any case, for arbitrary coefficients R we have

$$H^{\mathbb{Z}_2}_{i-\lambda_r}(B_r;R) = H_{i-\lambda_r}(B_r \times_{\mathbb{Z}_2} E\mathbb{Z}_2;R) \cong H_{i-\lambda_r}(B_r/\vartheta;R).$$

Alternatively, this follows directly from the Leray-Hirsch theorem ([**Hat02**, Theorem 4D.1] or [**Spa95**, Chapter 5, Section 7, Theorem 9] for the homology version) since the fibre is path-connected.

We now answer the question whether the bundles $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ are orientable: We can use [**Rad87**, Satz 5.9] according to which we inspect the $\mathbb{Z}_2$-action on the Thom class $\tau_r \in H^{(2r-1)(n-1)}(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Z})$ as follows: τ_r exists since $p_r : \Gamma_r^- \to B_r$ is orientable because B_r is simply connected. $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ is still a vector bundle but the base is not simply connected anymore: $\pi_1(B_r/\vartheta) \cong \mathbb{Z}_2$. We must therefore find out how the nontrivial element of $\pi_1(B_r/\vartheta)$ acts on the (co-)homology of the fibre $(\mathbb{R}^{(2r-1)(n-1)}, \mathbb{R}^{(2r-1)(n-1)} - \{0\})$. More precisely, let γ be a curve on B_r connecting the great circle c to $\bar{c} = \vartheta(c)$. γ is not a closed curve but projects to a loop $\tilde{\gamma}$ in B_r/ϑ. What is the action of $\tilde{\gamma}$ on τ_r?
Let us also use τ_r to denote the corresponding class in loop space cohomology

$$\tau_r \in H^{(2r-1)(n-1)}(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Z}) \cong H^{(2r-1)(n-1)}(\Lambda S^{n \le 2\pi^2 r}, \Lambda S^{n < 2\pi^2 r};\mathbb{Z}) \cong H^{(2r-1)(n-1)}(\Lambda S^n;\mathbb{Z})$$

where one should not forget that this holds since $M = S^n$: For spheres the level homology classes survive and the inclusion is equivariant.

We will now prove

PROPOSITION 3.4. *Let $\tau_r \in H^{(2r-1)(n-1)}(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Z})$ be Thom class of the negative normal bundle $p_r : \Gamma_r^- \to B_r$ of the submanifold $B_r \subset \Lambda S^n$ of r-fold covered great circles. Here $p_r : \Gamma_r^- \to B_r$ is the maximal subbundle of the normal bundle of $B_r \subset \Lambda S^n$ on which the Hessian $dE^2_{g_{st}}$ is negative definite.*
We have

$$\vartheta_*(\tau_r) = \begin{cases} \tau_r, & \text{if } n \text{ and } r \text{ are even} \\ -\tau_r, & \text{if } n \text{ or } r \text{ is odd} \end{cases}.$$

From this Proposition, together with the above Lemma, it follows that the bundle $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ is thus orientable if and only if n and r are even. The proof of the Proposition is given on the next few pages.

Let $p \in S^n$ and consider the space $\Omega_p S^n = \Omega S^n$ of loops based at p. Since ΩS^n is an H-space, even an H-group, its singular homology carries a Pontrjagin product. Let $\star$ denote this Pontrjagin product. It is well known that for $n \ge 2$ (with $\mathbb{Z}$-coefficients) the Pontrjagin ring of spheres is a polynomial ring: $(H_*(\Omega S^n;\mathbb{Z}), \star) \cong \mathbb{Z}[x]$ with $x \in H_{n-1}(\Omega S^n;\mathbb{Z})$ (see e.g. [**Hat02**, Discussion at the beginning of Section 4J, mainly Lemma 4J.2]).

LEMMA 3.5. *Let ϑ be the orientation-reversal of loops on the based loop space ΩS^n of S^n with $n \ge 2$. Then, for the generator $x \in H_{n-1}(\Omega S^n;\mathbb{Z})$ of the Pontrjagin ring $(H_*(\Omega S^n;\mathbb{Z}), \star) \cong \mathbb{Z}[x]$, we have*

$$\vartheta_*(x) = -x.$$

PROOF. Consider $[id_{S^n}] \in \pi_n(S^n) = [S^n, S^n]_0 \cong [\Sigma S^{n-1}, S^n]_0$. Here $[X,Y]_0$ denotes the set of homotopy classes of base point preserving maps from a pointed space $X = (X, x_0)$ to a pointed space $Y = (Y, y_0)$ and ΣS^{n-1} denotes the reduced suspension of S^{n-1}. We have a homeomorphism $S^n \cong \Sigma S^{n-1}$ where a point in S^n is given by $[p,t]$ where $p \in S^{n-1}$ and $t \in [0,1]$ and $S^{n-1} \hookrightarrow S^n$ is the embedding of an equator containing the base point: $p \mapsto [p, \frac{1}{2}]$. The set $[\Sigma S^{n-1}, S^n]_0$ carries a group structure induced from the H-cogroup group structure on ΣS^{n-1} ([**Bre93**, Chapter VII, Definition 3.2 and Theorem 3.3]). The assignment

$$[\Sigma S^{n-1}, S^n]_0 \to [S^{n-1}, \Omega S^n]_0,\ [f] \mapsto [\hat{f}],$$

where

$$\hat{f}(p)(t) := f\big([p,t]\big)$$

is a group isomorphism ([**Bre93**, Chapter VII, Theorem 4.2]). Its inverse is $[g] \mapsto [\check{g}]$, where $\check{g}\big([p,t]\big) := g(p)(t)$. That is

$$\mathbb{Z} \cong \pi_n(S^n) \cong \pi_{n-1}(\Omega S^n)$$

as abelian groups for $n \geq 1$.
Define $x := \widehat{id_{S^n}}_*([S^{n-1}]) \in H_{n-1}(\Omega S^n; \mathbb{Z})$, where $[S^{n-1}] \in H_{n-1}(S^{n-1}; \mathbb{Z})$ is a generator. We now show that x generates $H_{n-1}(\Omega S^n; \mathbb{Z})$ and hence the whole algebra $(H_*(\Omega S^n; \mathbb{Z}), \star)$: Consider the evaluation at time $\frac{1}{2}$, $ev_{\frac{1}{2}} : \Omega S^n \to S^n$. We have $ev_{\frac{1}{2}} \circ \widehat{id_{S^n}} = id_{S^{n-1}} : S^{n-1} \to S^{n-1} \subset S^n;\ p \mapsto (t \mapsto [p,t]) \mapsto [p, \frac{1}{2}] \cong p$. It therefore follows that

$$H_{n-1}(S^{n-1}; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\widehat{id_{S^n}}_*} H_{n-1}(\Omega S^n; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{ev_{\frac{1}{2}*}} H_{n-1}(S^{n-1}; \mathbb{Z}) \cong \mathbb{Z}$$

is the identity and so $\widehat{id_{S^n}}_*$ must be multiplication by 1 or -1 and in particular is surjective and thus maps generator to generator.
$\vartheta : \Omega S^n \to \Omega S^n$, $\gamma \mapsto \bar{\gamma}$ is in fact the homotopy inversion of the H-group ΩS^n (see e.g. [**AGP02**, Lemma 2.8.5]). We thus have that $\vartheta_* : [S^{n-1}, \Omega S^n]_0 \to [S^{n-1}, \Omega S^n]_0$ is the inversion. Let $h : [S^{n-1}, \Omega S^n]_0 \to H_{n-1}(\Omega S^n)$, $[f] \mapsto f_*([S^{n-1}])$, denote the Hurewicz (group) homomorphism ([**Bre93**, Chapter VII, Definition 10.1]). The commutative diagram

$$\begin{array}{ccc} \pi_{n-1}(\Omega S^n) = [S^{n-1}, \Omega S^n]_0 & \xrightarrow{\vartheta_*} & [S^{n-1}, \Omega S^n]_0 = \pi_{n-1}(\Omega S^n) \\ \downarrow h & & \downarrow h \\ H_{n-1}(\Omega S^n; \mathbb{Z}) & \xrightarrow{\vartheta_*} & H_{n-1}(\Omega S^n; \mathbb{Z}) \end{array}$$

shows that $\vartheta_* : H_{n-1}(\Omega S^n; \mathbb{Z}) \to H_{n-1}(\Omega S^n; \mathbb{Z}) \cong \mathbb{Z}$ also sends an element to its (group) inverse, hence it is multiplication by -1. In particular,

$$\vartheta_*(x) = -x.$$

In fact $\vartheta_*(x) = \vartheta_*\big(h(\widehat{id_{S^n}})\big) = h\big(\vartheta_*(\widehat{id_{S^n}})\big) = h\big((\widehat{id_{S^n}})^{-1}\big) = h(-\widehat{id_{S^n}}) = -h(\widehat{id_{S^n}}) = -x.$

□

REMARK 3.6. Note that the Hurewicz theorem directly shows that $x \in H_{n-1}(\Omega S^n; \mathbb{Z})$ is a generator if $n \geq 3$: Either the long exact homotopy sequence ([**Hat02**, Theorem 4.41]) of the Serre fibration $\Omega S^n \to \Lambda S^n \xrightarrow{ev_0} S^n$ or simply $\pi_k(\Omega S^n) = \pi_{k+1}(S^n)$ show that ΩS^n is $(n-2)$-connected. The Hurewicz theorem ([**Bre93**, Chapter VII, Corollary 10.8]) thus says that for $n \geq 3$ the Hurewicz homomorphism $h : \pi_{n-1}(\Omega S^n) \to H_{n-1}(\Omega S^n; \mathbb{Z})$ is an isomorphism.

Consider the inclusion $j : \Omega M \hookrightarrow \Lambda M$. The loop space fibration $\Omega S^n \to \Lambda S^n \to S^n$ implies that $j_* : H_{n-1}(\Omega) \to H_{n-1}(\Lambda)$ is an isomorphism (see below). Also, $\Omega M \subset \Lambda M$ is a codimension-n submanifold and thus an umkehr map $j_!$ exists (see Chapter 2). Moreover $j_! : (H_*(\Lambda M), *) \to (H_{*-n}(\Omega M), \star)$ is an algebra homomorphism (see Section 3 Chapter 2). These previously established facts are now going to be used.
Lets us see how ϑ_* comports with the Pontrjagin multiplication of the generators of $H_*(\Omega S^n)$: Similar to the case of free loops, for $\Omega = \Omega M$ we have

$$\begin{tikzcd}
\Omega \times \Omega \arrow[r, "\phi"'] \arrow[d, "\mathrm{T}\circ(\vartheta\times\vartheta)"'] & \Omega \arrow[d, "\vartheta"'] \\
\Omega \times \Omega \arrow[r, "\phi"'] & \Omega
\end{tikzcd}$$

commutes (compare (4.8) of Chapter 3). Here $\mathrm{T} : \Omega \times \Omega \to \Omega \times \Omega, (\alpha, \beta) \mapsto (\beta, \alpha)$. Hence, we get the commutative diagram (any coefficients)

$$\begin{tikzcd}
H_i(\Omega) \otimes H_j(\Omega) \arrow[r, "\times"'] \arrow[d, "\vartheta_*\otimes\vartheta_*"'] & H_{i+j}(\Omega \times \Omega) \arrow[r, "\phi_*"'] \arrow[d, "(\vartheta\times\vartheta)_*"'] & H_{i+j}(\Omega) \arrow[dd, "\vartheta_*"'] \\
H_i(\Omega) \otimes H_j(\Omega) \arrow[r, "\times"'] \arrow[d] & H_{i+j}(\Omega \times \Omega) \arrow[d, "\mathrm{T}_*"'] & \\
H_j(\Omega) \otimes H_i(\Omega) \arrow[r, "(-1)^{ij}\times"'] & H_{i+j}(\Omega \times \Omega) \arrow[r, "\phi_*"'] & H_{i+j}(\Omega),
\end{tikzcd}$$

i.e. $(-1)^{ij}\vartheta_*(b) \star \vartheta_*(a) = \vartheta_*(a \star b)$ for all $a \in H_i(\Omega), b \in H_j(\Omega)$.
Now back to S^n: Since if $M = S^n$ the Pontrjagin algebra with integer coefficients is a polynomial algebra and thus in particular commutative, we can also write

$$(-1)^{ij}\vartheta_*(a) \star \vartheta_*(b) = \vartheta_*(a \star b)$$

for $a \in H_i(\Omega S^n; \mathbb{Z}), b \in H_j(\Omega S^n; \mathbb{Z})$. For any class $a \in H_i(\Omega S^n; \mathbb{Z})$ we have $\vartheta_*(a) = \pm a$ as ϑ is an involution and $H_i(\Omega S^n; \mathbb{Z})$ is either isomorphic to $\mathbb{Z}$ or 0. Thus if above $a = b$ we have

$$\vartheta_*(a^2) = \begin{cases} \big(\vartheta_*(a)\big)^2 = a^2, & \text{if } |a| \text{ is even} \\ -\big(\vartheta_*(a)\big)^2 = -a^2, & \text{if } |a| \text{ is odd} \end{cases}$$

and, more generally, for $k \geq 1$ we have

$$(-1)^{|a||a^{k-1}|}\vartheta_*(a^{k-1})\star\vartheta_*(a) = \vartheta_*(a\star a^{k-1}) = \vartheta_*(a^k) = \vartheta_*(a^{k-1}\star a) = (-1)^{|a^{k-1}||a|}\vartheta_*(a)\star\vartheta_*(a^{k-1})$$

(even for a non-commutative Pontrjagin product) and so

$$\vartheta_*(a^k) = (-1)^{|a||a^{k-1}|}\vartheta_*(a) \star \vartheta_*(a^{k-1}) = \cdots = (-1)^{|a||a^{k-1}|} \cdot (-1)^{|a||a^{k-2}|} \ldots (-1)^{|a||a|}\vartheta_*(a)^k.$$

With $|a^k| = |a^{\star k}| = k \cdot |a|$ we get

$$(-1)^{|a||a^{k-1}|} \cdot (-1)^{|a||a^{k-2}|} \ldots (-1)^{|a||a|} = (-1)^{|a|^2 \cdot \sum_{i=1}^{k-1}(k-i)} = (-1)^{|a|^2 \cdot \frac{(k-1)k}{2}}.$$

Hence

$$\vartheta_*(a^k) = \begin{cases} (\vartheta_*(a))^k, & \text{if } |a| \equiv 0 \bmod 2 \text{ or } (k-1)k \equiv 0 \bmod 4 \\ -(\vartheta_*(a))^k, & \text{if } |a| \equiv 1 \bmod 2 \text{ and } (k-1)k \not\equiv 0 \bmod 4 \end{cases}.$$

The same analysis for the Chas-Sullivan product even yields

PROPOSITION 3.7. *Let M be a connected, compact manifold. Then, the homomorphism $\vartheta_* = \theta_* : \big(H_*(\Lambda M), *\big) \to \big(H_*(\Lambda M), *\big)$ is an algebra endomorphism: for all $a \in H_i(\Lambda), b \in H_j(\Lambda)$ we have*

$$\vartheta_*(a) * \vartheta_*(b) = \vartheta_*(a * b). \tag{3.3}$$

PROOF. This follows from the following commutative diagram (any coefficients):

$$\begin{tikzcd}
H_{i+j}(\Lambda^2, \Lambda^2 - \mathcal{F}) \arrow[r, "\cap(\vartheta\times\vartheta)^*(\tau_{\mathcal{F}})=\tau_{\mathcal{F}}"'] \arrow[d, "(\vartheta\times\vartheta)_*"'] & H_{i+j-n}(\mathcal{F}) \arrow[r, "\phi_*"'] \arrow[d, "(\vartheta\times\vartheta)_*"'] & H_{i+j-n}(\Lambda) \arrow[d, "\theta_*"'] \arrow[dd, bend left, "\vartheta_*"] \\
H_{i+j}(\Lambda^2, \Lambda^2 - \mathcal{F}) \arrow[r, "\cap\tau_{\mathcal{F}}"'] \arrow[d, "\mathrm{T}_*"'] & H_{i+j-n}(\mathcal{F}) \arrow[r, "\phi_*"'] \arrow[d, "\mathrm{T}_*"'] & H_{i+j-n}(\Lambda) \arrow[d, "(\chi_{\frac{1}{2}})_*=id"'] \\
H_{i+j}(\Lambda^2, \Lambda^2 - \mathcal{F}) \arrow[r, "(-1)^n\cap\tau_{\mathcal{F}}"'] & H_{i+j-n}(\mathcal{F}) \arrow[r, "\phi_*"'] & H_{i+j-n}(\Lambda).
\end{tikzcd}$$

This diagram commutes since

- For the $\vartheta\times\vartheta : \Lambda^2 \to \Lambda^2$ we have seen ((3.7) of Chapter 3) that $\big((\vartheta\times\vartheta)\times id_N\big)^*(\tau_{\mathcal{F}}) = \tau_{\mathcal{F}}$ holds. The above is a mild abuse of notation.
- Permuting the factors in $\Lambda \times \Lambda$ implies that an additional $(-1)^n$ in the lower left square is needed to make it commutative.

The upper part thus implies that

$$\vartheta_*(a) * \vartheta_*(b) = \theta_*(a * b)$$

holds, while the whole diagram leads to

$$(-1)^{ij+n}\vartheta_*(b) * \vartheta_*(a) = \vartheta_*(a * b)$$

for $a \in H_i(\Lambda), b \in H_j(\Lambda)$. Both equations yield (3.3), the first via $\vartheta_* = \theta_*$, the second via $(-1)^{ij+n}\vartheta_*(b) *' \vartheta_*(a) = \vartheta_*(a) *' \vartheta_*(b)$, where $*'$ is again the Chas-Sullivan product without sign-correction. Hence, we have

$$\vartheta_*(a^k) = \vartheta_*(a)^k.$$

$\square$

Let $\sigma_r \in H_{(2r-1)(n-1)}(\Lambda;\mathbb{Z}) \cong \mathbb{Z}$ be the designated generators. As explained in Section 4 of Chapter 2, we have $\sigma_r = \sigma_1 * \Theta^{r-1} = \Theta^{r-1} * \sigma_1$. Here $\Theta \in H_{3n-2}(\Lambda)$ is a designated generator (compare [**HR13**, Lemma 5.4]). The $\sigma_r \in H_{(2r-1)(n-1)}(\Lambda;\mathbb{Z}) \cong \mathbb{Z}$ are Kronecker dual to the $\tau_r \in H^{(2r-1)(n-1)}(\Lambda;\mathbb{Z}) \cong \mathbb{Z}$, i.e. $\tau_r(\sigma_r) = 1$: As $H_{(2r-1)(n-1)}(\Lambda;\mathbb{Z}) = \mathbb{Z}, 0$ if n is at least 3, we have $H^{(2r-1)(n-1)}(\Lambda;\mathbb{Z}) \cong Hom_{\mathbb{Z}}\big(H_{(2r-1)(n-1)}(\Lambda;\mathbb{Z}), \mathbb{Z}\big) \cong \mathbb{Z}$. We orient $p_r : \Gamma_r^- \to B_r$ such that $\tau_r(\sigma_r) = 1$ holds.

Also $j_*(x) = \sigma_1$. This follows from the long exact homotopy sequence associated to the Serre fibration $\Omega S^n \xrightarrow{j} \Lambda S^n \xrightarrow{ev_0} S^n$ ([**Hat02**, Theorem 4.41]). The sequence implies that

$$H_{n-1}(\Omega S^n;\mathbb{Z}) \cong \pi_{n-1}(\Omega S^n) \xrightarrow{j_*} \pi_{n-1}(\Lambda S^n) \cong\ H_{n-1}(\Lambda S^n;\mathbb{Z})$$

is surjective. Hence j_* maps generators to generators in degree $n-1$. The isomorphism $H_{n-1}(\Omega S^n;\mathbb{Z}) \cong \pi_{n-1}(\Omega S^n)$ and $\pi_{n-1}(\Lambda S^n) \cong\ H_{n-1}(\Lambda S^n;\mathbb{Z})$ also follow from this homotopy sequence: It implies that $0 = \pi_k(\Omega S^n) = \pi_k(\Lambda S^n)$ for $k \leq n-2$, i.e. that ΩS^n and ΛS^n are $(n-2)$-connected. This in turn implies, via the Hurewicz theorem, that the Hurewicz map is an isomorphism in degree $n-1$ if $n-1$ is at least 2 (see e.g. [**Hat02**, Theorem 4.32 or Theorem 4.37]).

We now use the relations between the algebras of the spaces in the fibration $\Omega \to \Lambda \to M$ of Proposition 3.3 of Chapter 1. For convenience we recall them here:

$$j_!(a * b) = j_!(a) \star j_!(b) \tag{3.4}$$

$$j_*(y) * a = j_*\big(y \star j_!(a)\big) \tag{3.5}$$

$$j_*\big(j_!(a)\big) = A * a \tag{3.6}$$

where $A \in H_0(\Lambda)$ is a generator. Using these for $M = S^n$ we obtain:

- for n odd we have $U^{*2} = \Theta$, where $U \in H_{2n-1}(\Lambda S^n; \mathbb{Z})$ is a generator. Hence

$$\begin{aligned}
\sigma_r &= \sigma_1 * \Theta^{*(r-1)} = \sigma_1 * U^{*(2r-2)} \\
&= A * U^{*(2r-1)} \text{ since } \sigma_1 = A * U \\
&= j_* j_!(U^{*(2r-1)}) \text{ by equation 3.6} \\
&= j_* j_!(U)^{\star(2r-1)} \text{ by equation 3.4} \\
&= j_*(x^{\star(2r-1)}) \text{ since } j_!(U) = x
\end{aligned}$$

(compare [**HR13**, Section 6]). Note that $j_!(U) = x$ must hold, as the above for $r = 1$ is $j_*(x) = \sigma_1 = j_*\big(j_!(U)\big)$. Since $|x| = n-1$ is even, it follows that

$$\begin{aligned}
\vartheta_*(\sigma_r) &= \vartheta_*\big(j_*(x^{\star(2r-1)})\big) = j_*\big(\vartheta_*(x^{\star(2r-1)})\big) = j_*\big(\vartheta_*(x)^{\star(2r-1)}\big) \\
&= j_*(-x^{\star(2r-1)}) = -j_*(x^{\star(2r-1)}) \\
&= -\sigma_r.
\end{aligned}$$

Hence, via $\sigma_r(\tau_r) = 1 = \big((\vartheta_* \circ \vartheta_*)(\sigma_r)\big)(\tau_r) = \vartheta_*(\sigma_r)\big(\vartheta^*(\tau_r)\big) = -\sigma_r\big(\vartheta^*(\tau_r)\big)$, we get

$$\vartheta^*(\tau_r) = -\tau_r\,.$$

So if n is odd, the vector bundle $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ is not orientable.

- for n even we only get

$$\begin{aligned}
\sigma_r &= \sigma_1 * \Theta^{*(r-1)} = j_*(x) * \Theta^{*(r-1)} \\
&= j_*\big(x \star j_!(\Theta^{*(r-1)})\big) \text{ by equation 3.5} \\
&= j_*\big(x \star j_!(\Theta)^{\star(r-1)}\big) \text{ by equation 3.4}
\end{aligned}$$

since $H_{2n-1}(\Lambda S^n; \mathbb{Z}) = 0$ (compare [**HR13**, Section 6]). Hence we have

$$\begin{aligned}
\vartheta_*(\sigma_r) &= \vartheta_*\Big(j_*\big(x \star j_!(\Theta)^{\star(r-1)}\big)\Big) = j_*\Big(\vartheta_*\big(x \star j_!(\Theta)^{\star(r-1)}\big)\Big) \\
&= j_*\Big(\vartheta_*(x) \star \vartheta_*\big(j_!(\Theta)^{\star(r-1)}\big)\Big) = -j_*\Big(x \star \vartheta_*\big(j_!(\Theta)^{\star(r-1)}\big)\Big) \\
&= -j_*\Big(x \star \big(\vartheta_* \circ j_!(\Theta)\big)^{\star(r-1)}\Big),
\end{aligned}$$

since $j_!(\Theta)$ has even degree. $\sigma_r = j_*(x \star j_!(\Theta)^{*(r-1)})$ being a generator implies that $j_!(\Theta) = \pm x^2$. Thus

$$\begin{aligned}\vartheta_*\big(j_!(\Theta)\big) &= \vartheta_*(\pm x^{\star 2}) = \pm\vartheta_*(x^{\star 2}) = \pm(-x^{\star 2}) = -1(\pm x^{\star 2})\\ &= -j_!(\Theta)\end{aligned}$$

and so

$$\vartheta_*(\sigma_r) = (-1)^r j_*\Big(x \star \big(j_!(\Theta)\big)^{\star(r-1)}\Big) = (-1)^r \sigma_r.$$

This concludes the proof of Proposition 3.4 and shows

COROLLARY 3.8. *The bundle $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ is orientable if and only if n and r are even (compare* [**Rad87**, Satz 5.9]*).*

PROOF. This is Lemma 3.1 together with Proposition 3.4. □

REMARK 3.9. In fact $\vartheta_*(\Theta) = (-1)^{(n-1)}\Theta$:

- If n is odd we have $\vartheta_*(\Theta) = \vartheta_*(U^{*2}) = \vartheta_*(U)^{*2}$ (see proposition 3.7) and $-A * U = -\sigma_1 = \vartheta_*(\sigma_1) = \vartheta_*(A * U) = \vartheta_*(A) * \vartheta_*(U) = A * \vartheta_*(U)$. Since the homology is torsion-free for n odd and with at most one generator in each degree, it follows that $\vartheta_*(U) = -U$ and hence $\vartheta_*(\Theta) = \Theta$ if n is odd.

- If n is even $\vartheta_*(\sigma_2) = \vartheta_*(\sigma_1 * \Theta) = \vartheta_*(\sigma_1) * \vartheta_*(\Theta) = -\sigma_1 * \vartheta_*(\Theta)$ (again by equation 3.3). But we have just shown that $\vartheta_*(\sigma_2) = \sigma_2$ hence $\vartheta_*(\Theta) = -\Theta$ for even n.

3.2. Computation of $H_*(\Lambda S^n/\vartheta;\mathbb{Q})$. We already have enough information to compute the rational homology of $\Lambda S^n/\vartheta$:

- From Section 2 of Chapter 3 we know that $H_*(\Lambda/\vartheta;\mathbb{Q})$ is isomorphic to the subspace of $H_*(\Lambda;\mathbb{Q})$ consisting of those classes which are fixed under ϑ_*. Hence, for a class $x \in H_i(\Lambda;\mathbb{Q})$ the image $q_*(x) \in H_i(\Lambda/\vartheta;\mathbb{Q})$ of x under the surjective homomorphism q_* induced by $q : \Lambda \to \Lambda/\vartheta$ is nonzero if and only if $\vartheta_*(x) = x$.
- We know how ϑ_* acts on the generators of the loop homology algebra of spheres, namely:
  - For n odd the Chas-Sullivan algebra $\big(H_*(\Lambda S^n;\mathbb{Z});*\big)$ has two generators A and U and a unit E (see Section 4 of Chapter 2) and we have
$$\begin{aligned}\vartheta_*(A) &= A,\\ \vartheta_*(E) &= E,\\ \vartheta_*(U) &= -U.\end{aligned}$$
  - for n even the Chas-Sullivan algebra $\big(H_*(\Lambda S^n;\mathbb{Z});*\big)$ has three generators A, σ_1 and Θ and a unit E (see section of Chapter 1) and we have
$$\begin{aligned}\vartheta_*(A) &= A,\\ \vartheta_*(E) &= E,\\ \vartheta_*(\sigma_1) &= -\sigma_1,\\ \vartheta_*(\Theta) &= -\Theta.\end{aligned}$$

  Note that is is clear that A and E are mapped to themselves since on the base manifold M of ΛM the orientation reversal is the trivial action.

- The universal coefficient theorem for homology ([**Hat02**, Theorem 3A.3]) shows, since $\mathbb{Q}$ is torsionfree, that $H_i(\Lambda S^n;\mathbb{Q}) \cong H_i(\Lambda S^n;\mathbb{Z})\otimes\mathbb{Q}$ is either zero or isomorphic to $\mathbb{Q}$ since $H_i(\Lambda S^n;\mathbb{Z})$ is either 0 or $\mathbb{Z}$ or $\mathbb{Z}_2$, as we have seen in Section 4 of Chapter 2. The Chas-Sullivan product is defined in exact the same way as for $\mathbb{Z}$-coefficients and all classes of $H_*(\Lambda S^n;\mathbb{Q})$ can be written as a Chas-Sullivan product of the rational classes corresponding to the generators introduced in Section 4 of Chapter 2 (compare [**HR13**, Lemma 5.4]). The only difference is that for the even-dimensional spheres the product $A * \Theta^{*k} = 0$ in rational homology for all $k \in \mathbb{N}$ as these are torsion classes in $H_*(\Lambda S^n;\mathbb{Z})$.
  We denote rational generators by the same names, and we know how they behave under the homomorphism ϑ_*:
  - For n odd the Chas-Sullivan algebra $\big(H_*(\Lambda S^n;\mathbb{Q});*\big)$ has two generators $A \in H_0(\Lambda S^n;\mathbb{Q})$ and $U \in H_{2n-1}(\Lambda S^n;\mathbb{Q})$ and a unit $E \in H_n(\Lambda S^n;\mathbb{Q})$ and we have
  $$\begin{aligned}\vartheta_*(A) = A &\Rightarrow q_*(A) \neq 0,\\ \vartheta_*(E) = E &\Rightarrow q_*(E) \neq 0,\\ \vartheta_*(U) = -U &\Rightarrow q_*(U) = 0.\end{aligned}$$
  - for n even the Chas-Sullivan algebra $\big(H_*(\Lambda S^n;\mathbb{Q});*\big)$ has three generators $A \in H_0(\Lambda S^n;\mathbb{Q})$, $\sigma_1 \in H_{n-1}(\Lambda S^n;\mathbb{Q})$ and Θ and a unit $E \in H_n(\Lambda S^n;\mathbb{Q})$ and we have
  $$\begin{aligned}\vartheta_*(A) = A &\Rightarrow q_*(A) \neq 0,\\ \vartheta_*(E) = E &\Rightarrow q_*(E) \neq 0,\\ \vartheta_*(\sigma_1) = -\sigma_1 &\Rightarrow q_*(\sigma_1) = 0,\\ \vartheta_*(\Theta) = -\Theta &\Rightarrow q_*(\Theta) = 0.\end{aligned}$$
- Since we also know that $\vartheta_*(x * y) = \vartheta_*(x) * \vartheta_*(y)$ for all $x, y \in H_*(\Lambda S^n;\mathbb{Q})$ (Proposition 3.7), it suffices to know how ϑ_* acts on the generators to compute $H_*(\Lambda/\vartheta;\mathbb{Q})$.

Nevertheless, we are going to give some slightly different way of computing $H_*(\Lambda/\vartheta;\mathbb{Q})$, providing some insight into the origin of the loop homology classes: From [**Rad87**, pages 91-92] we get a sort of Thom isomorphism also for the unorientable case, i.e. when $\vartheta^*(\tau_r) = -\tau_r$, at least if we are using rational coefficients:

LEMMA 3.10. *Let $B_r \subset \Lambda S^n$ be the submanifold of r-fold covered great circles and let $p_r : \Gamma_r^- \to B_r$ denote its negative normal bundle with respect to the standard metric. Let $H_i(B_r;\mathbb{Q})^{\vartheta}$ denote the submodule of classes which are fixed under the homomorphism $\vartheta_* : H_i(B_r;\mathbb{Q}) \to H_i(B_r;\mathbb{Q})$. Then we have*

- *that*
  $$H_i(\Gamma_r^-/\vartheta, \Gamma_r^-/\vartheta - B_r/\vartheta;\mathbb{Q}) \cong H_{i-\lambda_r}(B_r/\vartheta;\mathbb{Q})$$
  if $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ is orientable
- *and that*
  $$H_i(\Gamma_r^-/\vartheta, \Gamma_r^-/\vartheta - B_r/\vartheta;\mathbb{Q}) \cong H_{i-\lambda_r}(B_r;\mathbb{Q})/H_{i-\lambda_r}(B_r;\mathbb{Q})^{\vartheta}$$
  if $p_r/\vartheta : \Gamma_r^-/\vartheta \to B_r/\vartheta$ is not orientable.

PROOF. Using the coefficient field $\mathbb{Q}$ we can employ the following fact ([**Hat02**, Proposition 3G.1] or [**Bre72**, Chapter III, Theorem 2.4] or [**Smi83**]): The two-fold covering map $p : B_r \to B_r/\vartheta$ induces an isomorphism

$$p_* : H_i(B_r;\mathbb{Q})^{\vartheta} \xrightarrow{\cong} H_i(B_r/\vartheta;\mathbb{Q})$$

where $H_i(B_r)^{\vartheta} := \{a \in H_i(B_r)|\, \vartheta_*(a) = a\} = \ker(\vartheta_* - id)$. In particular $p_* : H_i(B_r;\mathbb{Q}) \to H_i(B_r/\vartheta;\mathbb{Q})$ is surjective.
Let $\Phi_r : H_i(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Q}) \to H_{i-\lambda_r}(B_r;\mathbb{Q})$ be the Thom isomorphism given by $\Phi_r(x) = p_{r*}(\tau_r \cap x)$ (c.f. [**Spa95**, Chapter 5, Section 7, Theorem 10]). We then have

$$\Phi_r\big(\vartheta_*(x)\big) = p_{r*}\big(\tau_r \cap \vartheta_*(x)\big) = p_{r*}\big(\vartheta_*(\vartheta^*(\tau_r) \cap x)\big) = p_{r*}\big(\vartheta_*(-\tau_r \cap x)\big) = -\vartheta_* \circ \Phi_r(x),$$

due to the naturality of the cap product and since p_r is clearly equivariant. Consider the quotient map $\pi : H_{i-\lambda_r}(B_r;\mathbb{Q}) \to H_{i-\lambda_r}(B_r;Q)/H_{i-\lambda_r}(B_r;Q)^{\vartheta}$. Since we are using $\mathbb{Q}$ as coefficient field, 2 is invertible and hence

$$H_i(B_r;\mathbb{Q}) \cong \{a|\ \vartheta_*(a) = a\} \oplus \{a|\ \vartheta_*(a) = -a\} = H_i(B_r;\mathbb{Q})^{\vartheta} \oplus \{a|\ \vartheta_*(a) = -a\}.$$

The following diagram is then commutative:

$$\begin{tikzcd}
H_i(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Q}) \arrow[r, "\Phi_r"] \arrow[d, "-\vartheta_*"'] & H_{i-\lambda_r}(B_r;\mathbb{Q}) \arrow[r, "\pi"] \arrow[d, "\vartheta_*"'] & H_{i-\lambda_r}(B_r;\mathbb{Q})/H_{i-\lambda_r}(B_r;\mathbb{Q})^{\vartheta} \cong \{a|\ \vartheta_*(a) = -a\} \arrow[d, "-id"'] \\
H_i(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Q}) \arrow[r, "\Phi_r"] & H_{i-\lambda_r}(B_r;\mathbb{Q}) \arrow[r, "\pi"] & H_{i-\lambda_r}(B_r;\mathbb{Q})/H_{i-\lambda_r}(B_r;\mathbb{Q})^{\vartheta} \cong \{a|\ \vartheta_*(a) = -a\}.
\end{tikzcd}$$

and hence shows that $\pi \circ \Phi_r = \pi \circ \Phi_r \circ \vartheta_*$. We have

$$\begin{aligned}
\ker(\pi \circ \Phi_r) &= {\Phi_r}^{-1}\big(\ker(\pi)\big) = {\Phi_r}^{-1}\big(\{a \in H_i(B_r;\mathbb{Q})|\ \vartheta_*(a) = a\}\big) \\
&= \{{\Phi_r}^{-1}(a) \in H_i(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Q})|\ {\Phi_r}^{-1}\big(\vartheta_*(a)\big) = {\Phi_r}^{-1}(a)\} \\
&= \{{\Phi_r}^{-1}(a)|\ -\vartheta_*\big({\Phi_r}^{-1}(a)\big) = {\Phi_r}^{-1}(a)\} \\
&= \{x \in H_i(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Q})|\ \vartheta_*(x) = -x\} \\
&= \ker(id + \vartheta_*).
\end{aligned}$$

Since $(\Gamma_r^-, \Gamma_r^- - B_r) \to (\Gamma_r^-/\vartheta, \Gamma_r^-/\vartheta - B_r/\vartheta)$ is a two-fold covering, we again have (by the five-lemma or directly by [**Bre72**, Chapter III, Theorem 2.4])

$$\begin{aligned}
H_i(\Gamma_r^-, \Gamma_r^- - B_r;\mathbb{Q}) &\cong \ker(id - \vartheta_*) \oplus \ker(id + \vartheta_*), \\
\ker(id - \vartheta_*) &\cong H_i(\Gamma_r^-/\vartheta, \Gamma_r^-/\vartheta - B_r/\vartheta;\mathbb{Q}).
\end{aligned}$$

Thus, setting the kernel of the surjective map $\pi \circ \Phi_r$ equal to zero, we get an isomorphism

$$\{a|\ \vartheta_*(a) = a\} \cong H_i(\Gamma_r/\vartheta, \Gamma_r - B_r/\vartheta;\mathbb{Q}) \xrightarrow{\cong} H_{i-\lambda_r}(B_r;\mathbb{Q})/H_{i-\lambda_r}(B_r;\mathbb{Q})^{\vartheta} \cong \{a|\ \vartheta_*(a) = -a\}.$$

This says that the Thom isomorphism induces an isomorphism between the eigenspace of ϑ_* to the eigenvalue 1 on the homology of Γ_r^- and the eigenspace of ϑ_* to the eigenvalue -1 on the homology of B_r. In contrast, in the orientable case it maps eigenspace of 1 to eigenspace of 1. $\square$

Hence we simply have to compute the homology of B_r/ϑ:

LEMMA 3.11. *Let $B_r \subset \Lambda S^n$ denote the r-fold covered great circles. For any $r \in \mathbb{N}$ we have*

- *for n odd*

$$H_i(B_r/\vartheta;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = n \\ \mathbb{Q} & i = 0 \\ \{0\} & \textit{otherwise} \end{cases}$$

- *and for even n*

$$H_i(B_r/\vartheta;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 2n-1 \\ \mathbb{Q} & i = 0 \\ \{0\} & \textit{otherwise} \end{cases}$$

Proof. We have (see e.g. [**Bre93**, Chapter VI, Example 13.5])

- for n odd

$$H_i(B_r;\mathbb{Q}) \cong H_i(T^1S^n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 2n-1 \\ \mathbb{Q} & i = n \\ \mathbb{Q} & i = n-1 \\ \mathbb{Q} & i = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

- and for even n

$$H_i(B_r;\mathbb{Q}) \cong H_i(T^1S^n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 2n-1 \\ \mathbb{Q} & i = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

for all r. Recall that $\vartheta(p,v) = (p,-v)$ for $(p,v) \in V_2(\mathbb{R}^{n+1}) \cong T^1S^n \cong B_r$ with equivariant homeomorphisms $V_2(\mathbb{R}^{n+1}) \cong T^1S^n \cong B_r$. It follows that B_r is the total space of a sphere bundle over S^n: $S^{n-1} \to B_r/\vartheta \to S^n$. ϑ is fibrewise the antipodal map. The Serre spectral sequence of the induced fibration $\mathbb{R}P^{n-1} \to B_r/\vartheta \to S^n$ immediately shows that

- for n odd

$$H_i(B_r/\vartheta;\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = n \\ \mathbb{Q} & i = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

- and for even n

$$H_i(B_r/\vartheta;\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 2n-1 \\ ? & i = n \\ ? & i = n-1 \\ \mathbb{Q} & i = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

  To find out what the question marks are, we have to know the n^{th} transgression map, i.e. the differential $d^n_{0,n-1} : E^n_{n,0} \to E^n_{0,n-1}$, which is likely to be equal to multiplication by 2.

To get the full answer we can alternatively again use $p_* : H_i(B_r;\mathbb{Q})^\vartheta \xrightarrow{\cong} H_i(B_r/\vartheta;\mathbb{Q})$: Looking again at the fibration $S^{n-1} \to B_r \cong T^1S^n \xrightarrow{f} S^n$, the above reflects that

- for n odd the generators $g_{n-1} \in H_{n-1}(B_r;\mathbb{Z})$ and $g_{2n-1} \in H_{2n-1}(B_r;\mathbb{Z})$ are mapped to their inverses under ϑ_*: This holds since the bundle projection f has a section if n is odd. It follows that f_* is a surjective homomorphism. The long exact homotopy sequence of the fibration thus breaks up into short exact sequences and together with the naturality of the Hurewicz map h we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n-1}(S^{n-1}) \arrow[r, "i_*"] \arrow[d, "h"', "\cong"] & \pi_{n-1}(B_r) \arrow[r, "f_*"] \arrow[d, "h"', "\cong"] & \pi_{n-1}(S^n) = 0 \arrow[r] \arrow[d, "h"', "\cong"] & 0 \\
0 \arrow[r] & H_{n-1}(S^{n-1};\mathbb{Z}) \arrow[r, "i_*"] & H_{n-1}(B_r;\mathbb{Z}) \arrow[r, "f_*"] & H_{n-1}(S^n;\mathbb{Z}) = 0 \arrow[r] & 0
\end{tikzcd}$$

where i denotes the inclusion of the fibre S^{n-1}. Note that h is an isomorphism here since all the spaces involved are $(n-2)$-connected. This shows that i_* is an isomorphism in degree $n-1$ and thus $g_{n-1} = i_*([S^{n-1}])$. As the action is fibrewise, $\vartheta(p,v) = (p,-v)$, it follows that ϑ_* and i_* commute. ϑ is the antipodal map on the fibres and since $n-1$ is even, ϑ reverses the orientation on the fibre and hence $\vartheta_*(g_{n-1}) = (\vartheta_* \circ i_*)([S^{n-1}]) = i_*(-[S^{n-1}]) = -g_{n-1}$.
For the top homology generator $g_{2n-1} = [B_r]$ a local degree calculation shows $\vartheta_*([B_r]) = -[B_r]$: We have $H_i(B_r) \cong H_i(S^n) \oplus H_{i-n+1}(S^n)$ from the Gysin sequence and $H_i(X) \oplus H_{i-n+1}(X) = \bigoplus_{k+j=i} H_k(X) \otimes H_j(S^{n-1})$, for any space X, so that $H_*(B_r) \cong H_*(S^n) \otimes H_*(S^{n-1})$ as groups. On the other hand, the Künneth formula says $H_*(S^{n-1} \times S^n) \cong H_*(S^n) \otimes H_*(S^{n-1})$ and so we finally have $H_*(B_r) \cong H_*(S^{n-1} \times S^n)$, as if it was a product bundle. So, if we choose generators in top degree for S^n and S^{n-1} we have the correspondence $[S^n] \otimes [S^{n-1}] = [S^n \times S^{n-1}] = [B_r]$. Let $U \subset S^n$ be a trivializing open subset, so $U \times S^{n-1} \cong f^{-1}(U)$. We then have an

isomorphism k given by the following composition

$$
\begin{aligned}
k : H_n(S^n) \otimes H_{n-1}(S^{n-1}) & \\
&\xrightarrow[\text{Künneth Theorem}]{\cong} H_{2n-1}(S^n \times S^{n-1}) \\
&\xrightarrow[\text{orientability of } S^n \times S^{n-1}]{\cong} H_{2n-1}(S^n \times S^{n-1}, S^n \times S^{n-1} - \{(x,t)\}) \\
&\xrightarrow[\text{excision}]{\cong} H_{2n-1}(U \times S^{n-1}, U \times S^{n-1} - \{(x,t)\}) \\
&\xrightarrow{\cong} H_{2n-1}(f^{-1}(U), f^{-1}(U) - \{(x,t)\}) \\
&\xrightarrow[\text{excision}]{\cong} H_{2n-1}(B_r, B_r - \{(x,t)\}) \\
&\xrightarrow[\text{orientability of } B_r]{\cong} H_{2n-1}(B_r).
\end{aligned}
$$

Clearly, k commutes with ϑ_*:

$$
\begin{tikzcd}
H_n(S^n) \otimes H_{n-1}(S^{n-1}) \arrow[rr, "id\otimes\vartheta_* = id\otimes -id"] \arrow[d, "\cong"] \arrow[dd, dashed, bend right, "k"'] & & H_n(S^n) \otimes H_{n-1}(S^{n-1}) \arrow[d, "\cong"'] \arrow[dd, dashed, bend left, "k"] \\
H_{2n-1}(B_r, B_r - \{(x,t)\}) \arrow[rr, "\vartheta_*"] \arrow[d, "\cong"] & & H_{2n-1}(B_r, B_r - \{(x,-t)\}) \arrow[d, "\cong"'] \\
H_{2n-1}(B_r) \arrow[rr, "\vartheta_*"] & & H_{2n-1}(B_r).
\end{tikzcd}
$$

Thus $\vartheta_* = -id$ in top degree. In particular, B_r/ϑ is not orientable if n is odd.

Since f has a section, also the class $[S^n]$ from the base S^n injects into $H_*(B_r;\mathbb{Z})$. On the base S^n the action is trivial, hence $\vartheta_*([S^n]) = [S^n]$. n-dimensional homology of B_r thus survives in the quotient.
Since B_r is path connected, the inclusion of any point determines the same generator of H_0, i.e. $\vartheta_* = id$ in degree zero.

- for even n the same reasoning as above shows that $\vartheta_*(g_{2n-1}) = g_{2n-1}$ and the top class survives. Surjectivity of p_* implies that $H_n(B_r/\vartheta;\mathbb{Q})$ and $H_{n-1}(B_r/\vartheta;\mathbb{Q})$ are trivial. Thus the full homology survives:

$$H_i(B_r/\vartheta;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 2n-1 \\ \mathbb{Q} & i = 0 \\ \{0\} & \text{otherwise} \end{cases}.$$

□

The above considerations now give us the level homology via (3.1):

Proposition 3.12. *For the level homologies $H_i(\Lambda S^{n\le 2\pi^2 r}/\vartheta, \Lambda S^{n<2\pi^2 r}/\vartheta;\mathbb{Q})$ the following holds:*

- *If the bundle $\Gamma_r^-/\vartheta \to B_r/\vartheta$ is orientable we have*
$$\begin{aligned} H_i(\Lambda S^{n\le 2\pi^2 r}/\vartheta, \Lambda S^{n<2\pi^2 r}/\vartheta;\mathbb{Q}) &\cong H_i(\Gamma_r^-/\vartheta, (\Gamma_r^- - B_r)/\vartheta;\mathbb{Q}) \\ &\cong H_{i-\lambda_r}(B_r/\vartheta;\mathbb{Q}) \end{aligned}$$
- *and in the unorientable case we have*
$$\begin{aligned} H_i(\Lambda S^{n\le 2\pi^2 r}/\vartheta, \Lambda S^{n<2\pi^2 r}/\vartheta;\mathbb{Q}) &\cong H_i(\Gamma_r^-/\vartheta, (\Gamma_r^- - B_r)/\vartheta;\mathbb{Q}) \\ &\cong H_{i-\lambda_r}(B_r;\mathbb{Q})/H_{i-\lambda_r}(B_r;\mathbb{Q})^{\vartheta}. \end{aligned}$$

□

Remark 3.13. We have in fact also already computed the equivariant rational level homologies $H_i^{\mathbb{Z}_2}(\Lambda^{\le 2\pi^2 r}, \Lambda^{<2\pi^2 r};\mathbb{Q})$: As already mentioned before, since the action ϑ on Γ_r^- is free we have

$$H_i^{\mathbb{Z}_2}(\Gamma_r^-, \Gamma_r^- - B_r; R) = H_i(\Gamma_r^- \times_{\mathbb{Z}_2} E\mathbb{Z}_2, (\Gamma_r^- - B_r)\times_{\mathbb{Z}_2} E\mathbb{Z}_2; R) \cong H_i(\Gamma_r^-/\vartheta, (\Gamma_r^- - B_r)/\vartheta; R)$$

for arbitrary coefficients R, as $E\mathbb{Z}_2 \to \Gamma_r^- \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \to \Gamma_r^-/\vartheta$ is a fibre bundle (with contractible fibre) in that case. Combining this with equation (3.2) we get

$$\begin{aligned} H_i^{\mathbb{Z}_2}(\Lambda^{\le 2\pi^2 r}, \Lambda^{<2\pi^2 r}; R) &\cong H_i^{\mathbb{Z}_2}(D\Gamma_r^-, \partial D\Gamma_r^-; R) \\ &\cong H_i(D\Gamma_r^-/\vartheta, \partial D\Gamma_r^-/\vartheta; R) \\ &\cong H_i(\Gamma_r^-/\vartheta, (\Gamma_r^- - B_r)/\vartheta; R) \\ &\cong H_i(\Lambda^{\le 2\pi^2 r}/\vartheta, \Lambda^{<2\pi^2 r}/\vartheta; R) \end{aligned}$$

which then in particular holds for rational coefficients.

It is astonishing that with rational coefficients this even holds when the action of a finite group G is not free:

Proposition 3.14. *Let G be a finite discrete group acting continuously on a paracompact Hausdorff space X that is in addition G-equivariantly homotopy equivalent to a countable CW-complex. Then*

$$H_i^G(X;\mathbb{Q}) \cong H_i(X/G;\mathbb{Q}). \tag{3.7}$$

The assumptions are satisfied by the spaces $\Lambda, \Lambda^{\le a}$ (if a is a regular or non-degenerate critical value) and $\Lambda^{<a}$ ([**Rad87**, Theorem 5.5] or [**Rad89**, Theorem 4.2]).
We are not going to use this result since the actions under consideration are free on B_r, but it could turn out to useful for further considerations. For instance the actions of $\mathbb{Z}_m \subset S^1 \subset O(2)$ are not free, even on B_r for $r > 1$.

Proof using transfer. It suffices to assume that X has the homotopy type of a countable CW-complex, equivariance of the homotopy equivalence is not needed. As also EG is a countable CW-complex and we have isomorphisms given by the quotient maps (Property (3) of the section on transfers in Chapter 3):

$$\begin{aligned} H_i^G(X;\mathbb{Q}) &= H_i(X\times_G EG;\mathbb{Q}) \cong H_i(X\times EG;\mathbb{Q})^G,\\ H_i(X/G;\mathbb{Q}) &\cong H_i(X;\mathbb{Q})^G, \end{aligned}$$

where again $H_i(\cdot;\mathbb{Q})^G$ denotes the classes of degree i that are fixed under g_* for all $g\in G$. As EG is contractible the Künneth formula gives

$$H_i(X\times EG;\mathbb{Q}) \cong \bigoplus_k \big(H_{i-k}(X;\mathbb{Q})\otimes_{\mathbb{Q}} H_k(EG;\mathbb{Q})\big) = H_i(X;\mathbb{Q})\otimes_{\mathbb{Q}} H_0(EG;\mathbb{Q}) \cong H_i(X;\mathbb{Q})$$

where the first isomorphism is given by the homological cross product. It follows that for $y\in H_i(X\times EG;\mathbb{Q})$, given a generator $1\in H_0(EG;\mathbb{Q})$ we have $y=x\times 1$ for a unique $x\in H_i(X;\mathbb{Q})$. Then

$$(g\times g)_*(y) = g_*(x)\times g_*(1) = g_*(x)\times 1$$

since any continuous map induces the identity on $H_0(EG;\mathbb{Q})$ as EG is path-connected. Hence, y is fixed under G if and only if x is and so

$$H_i^G(X;\mathbb{Q}) \cong H_i(X\times EG;\mathbb{Q})^G \cong H_i(X;\mathbb{Q})^G \cong H_i(X/G;\mathbb{Q})$$

for all $i\in\mathbb{Z}$. □

Proof without using transfer. Let G be a finite group acting on a paracompact Hausdorff space X. Let $p_n : EG(n)\to BG(n)$ be the n-universal G-bundle. Consider the commutative diagram

$$\begin{tikzcd} X\times EG(n) \arrow[r, "pr_X"] \arrow[d] & X \arrow[d] \\ X\times_G EG(n) \arrow[r, "f_n:=pr_X/G"] & X/G. \end{tikzcd}$$

The fibres of f_n are homeomorphic to the n-classifying spaces of the stabilizer groups G_x of points x of X:

$$\begin{aligned} f_n^{-1}([x]) &= G\cdot x\times_G EG \cong G/G_x\times_G EG \cong \{pt\}\times_{G_x} G\times_G EG \cong \{pt\}\times_{G_x} EG\\ &\cong EG(n)/G_x \cong BG_x(n). \end{aligned}$$

This fibre $EG(n)/G_x$ is the base of the principal G_x bundle $p: EG(n)\to EG(n)/G_x$, which is therefore a finite covering. Hence $p_*: H_i(EG(n);\mathbb{Q})\to H_i(EG(n)/G_x;\mathbb{Q})$ is surjective and since $EG(n)$ is $(n-1)$-connected, $H_i(EG(n)/G_x;\mathbb{Q})=0$ for $0<i<n-1$. Moreover, since $EG(n)$ is path connected ($n>0$), we have

$$\tilde{H}_i(f_n^{-1}([x]);\mathbb{Q}) = 0$$

for $0\le i<n-1$ and for all $x\in X$. (Alternatively, the homotopy exact sequence for Serre fibrations gives $\pi_1(EG(n)/G_x)\cong G_x$ and $\pi_i(EG(n)/G_x)=0$ for $0<i<n-1$.)
Since $EG(n)$ is paracompact and Hausdorff and G is compact and Hausdorff $f_n : X\times_G EG(n)\to X/G$ is a map between paracompact Hausdorff spaces ([**Bre72**, Chapter I, Theorem 3.1] and [**Eng89**, Theorem 5.1.33]). In addition $f_n = pr_X/G$ is a closed map since $pr_X : X\times EG(n)\to X$ is closed because $EG(n)$ is compact ([**Eng89**, Theorem 3.7.1]).

Using the Vietoris-Begle mapping theorem ([**Spa95**, Chapter 6, Section 9, Theorem 15]) we deduce that

$$f_{n*} : H_i(X \times_G EG(n); \mathbb{Q}) \longrightarrow H_i(X/G; \mathbb{Q})$$

is an isomorphism for all n and all $0 \leq i < n-1$ (we may have to assume that the homology of X is finitely generated in each degree). This holds since the spaces $f_n^{-1}([x]) \cong EG(n)/G_x, X \times_G EG(n)$ and X/G are all homologically locally connected ([**Spa95**, Chapter 6, Section 9, Corollary 5]). $X \times_G EG(n)$ and X/G are homologically locally connected since any orbit in the G-spaces $X \times EG(n)$ and X possesses an open neighbourhood that G-equivariantly retracts to the orbit. This neighbourhood hence projects to a neighbourhood that retracts to a point in the quotient space ([**Bre72**, Chapter II, Sections 4 and 5], paracompact Hausdorff spaces are normal and hence completely regular). $EG(n)/G_x \cong BG_x(n)$ is even locally contractible since it has the homotopy type of a CW-complex ([**Mil56**, Section 5]).
Let $j^n_{n+1} : EG(n) \hookrightarrow EG(n+1)$ be the inclusion. Since it is G-equivariant so is $i^n_{n+1} := id_X \times j^n_{n+1} : X \times EG(n) \hookrightarrow X \times EG(n+1)$. Consider the commutative diagram

$$\begin{tikzcd}
X \times EG(n) \arrow[rr, "i^n_{n+1}"] \arrow[d] & & X \times EG(n+1) \arrow[d] \\
X \times_G EG(n) \arrow[rr, "\tilde{i}^n_{n+1}"] \arrow[dr, "f_n"'] & & X \times_G EG(n+1) \arrow[dl, "f_{n+1}"] \\
& X/G. &
\end{tikzcd}$$

In rational homology this yields the commutative diagram

$$\begin{tikzcd}
H_i(X \times_G EG(n); \mathbb{Q}) \arrow[rr, "\left(\tilde{i}^n_{n+1}\right)_*"] \arrow[dr, "f_{n*}", "\cong"'] & & H_i(X \times_G EG(n+1); \mathbb{Q}) \arrow[dl, "f_{n+1*}"', "\cong"] \\
& H_i(X/G; \mathbb{Q}). &
\end{tikzcd}$$

which shows that each $\left(\tilde{i}^n_{n+1}\right)_*$ is an isomorphism for $0 \leq i < n-1$. It follows that the uniquely determined homorphism $h := colim f_{n*} : colim H_i(X \times_G EG(n); \mathbb{Q}) \to H_i(X/G; \mathbb{Q})$ is an isomorphism for all i. Here *colim* denotes the colimit or direct limits of abelian groups ([**Bre93**, Appendix D]). This concept also exists for topological spaces, CW complexes for example have the colimit or union topology with respect to their subcomplexes ([**Dug78**, Appendix 2]).

The final step is to show that at least in homology we have $colim H_*(X \times_G EG(n); \mathbb{Q}) \cong H_*(X \times_G colim EG(n); \mathbb{Q}) = H_*(X \times_G EG; \mathbb{Q})$: The spaces X, EG and $EG(n)$ are compactly generated spaces. If follows that $X \times_k EG = colim\big(X \times_k EG(n)\big)$, where $\times_k$ denotes the compactly generated product ([**Ste67**, Definition 4.1 and Theorem 10.3]). Let Y be a countable CW-complex which is homotopy equivalent to X: $X \simeq Y$. Then, since EG is a countable CW-complex as well, the product $Y \times EG$ is a CW-complex with respect to the product cells and $Y \times EG = Y \times_k EG$ ([**Hat02**, Appendix, Theorem A.6]). In total

$$X \times EG \simeq Y \times EG = Y \times_k EG = colim\big(Y \times_k EG(n)\big) = colim\big(Y \times EG(n)\big) \simeq colim\big(X \times EG(n)\big).$$

Now since the homotopy equivalence $X \simeq Y$ is G-equivariant we get a homotopy equivalence

$$X \times_G EG \simeq colim\big(Y \times_G EG(n)\big) \simeq colim\big(X \times_G EG(n)\big).$$

By [**Hat02**, Chapter 3, Proposition 3.33], we have that $H_*\Big(colim\big(X \times_G EG(n)\big);\mathbb{Q}\Big) \cong colim H_*\big(X \times_G EG(n);\mathbb{Q}\big)$. Hence,

$$H_*(X/G;\mathbb{Q}) \cong colim H_*\big(X\times_G EG(n);\mathbb{Q}\big) \cong H_*\Big(colim\big(X\times_G EG(n)\big);\mathbb{Q}\Big) \cong H_*(X\times_G EG;\mathbb{Q}).$$

□

REMARK 3.15. A further fast proof of the proposition using the Leray spectral sequence of the map $f = pr_X/G : X \times_G EG \to X/G$ ([**Hsi75**, p.37]) should be possible.

We now compute $H_*(\Lambda/\vartheta;\mathbb{Q})$ by piecing the level homologies $H_*(\Lambda^{\leq 2\pi^2 r^2}/\vartheta, \Lambda^{<2\pi^2 r^2}/\vartheta;\mathbb{Q})$ together: Let us first define some generators of $H_*(\Lambda/\vartheta;\mathbb{Q})$. Let $q = q_\vartheta : \Lambda \to \Lambda/\vartheta$ be the quotient map. For both, n odd and even, the classes A and E coming from the point curves $\Lambda^0 \cong S^n$ are invariant under ϑ_* and so $q_{\vartheta *}(A)$ and $q_{\vartheta *}(E)$ are generators. Furthermore

- for n odd we have $\vartheta_*(U^{*k}) = (-1)^k U$ and hence

$$q_{\vartheta *}(U^{*k}) \neq 0 \Leftrightarrow k \text{ is even},$$
$$q_{\vartheta *}(A * U^{*k}) \neq 0 \Leftrightarrow k \text{ is even}.$$

  We make the following definition

$$\mu := q_{\vartheta *}(U^{*2}).$$

  Then μ is a generator of $H_{3n-2}(\Lambda/\vartheta;\mathbb{Q})$.

- for n even we have $\vartheta_*(\Theta^{*k}) = (-1)^k \Theta^{*k}$ and $\vartheta_*(\sigma_k) = (-1)^k \sigma_k$ and so

$$q_{\vartheta *}(\Theta^{*k}) \neq 0 \Leftrightarrow k \text{ is even},$$
$$q_{\vartheta *}(\sigma_k) = q_{\vartheta *}(\sigma_1 * \Theta^{*(k-1)}) \neq 0 \Leftrightarrow k \text{ is even}.$$

  We make the following definition

$$\eta := q_{\vartheta *}(\Theta^{*2}).$$

  Then η is a generator of $H_{5n-4}(\Lambda/\vartheta;\mathbb{Q})$.

PROPOSITION 3.16. *Let $\vartheta : \Lambda S^n \to \Lambda S^n$ be the involution that reverses the orientation of loops. Then, if $n \geq 3$, for the orbit space Λ/ϑ of the associated $\mathbb{Z}_2$-action we have:*

- *for n odd*

$$H_i(\Lambda S^n/\vartheta;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0, n \\ \mathbb{Q} & i = n-1+\lambda_r = 2r(n-1) \text{ for } r \in \mathbb{N} \\ \mathbb{Q} & i = 2n-1+\lambda_r = n+2r(n-1) \text{ for } r \in \mathbb{N} \\ \{0\} & \text{otherwise} \end{cases}$$

- *for n even*

$$H_i(\Lambda S^n/\vartheta;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0, n \\ \mathbb{Q} & i = \lambda_r = (2r-1)(n-1) \text{ for } r \in 2\mathbb{N} \\ \mathbb{Q} & i = 2n-1+\lambda_r = n+2r(n-1) \text{ for } r \in 2\mathbb{N} \\ \{0\} & \text{otherwise} \end{cases}$$

where $\lambda_r = (2r-1)(n-1)$ is the index of a r-fold iterated prime closed geodesic of the standard sphere (S^n, g_{st}).

We can plot this in a critical level (r) versus degree (d)-diagram. More precisely, we plot the level homologies in a r versus d-diagram. That is, the entry at the coordinate (r, d) is

$$H_d(\Lambda^{\leq 2\pi^2 r^2}/\vartheta, \Lambda^{<2\pi^2 r^2}/\vartheta;\mathbb{Q}) \cong H_d^{\mathbb{Z}_2}(\Lambda^{\leq 2\pi^2 r^2}, \Lambda^{<2\pi^2 r^2};\mathbb{Q}).$$

In turns out that the diagrams below shows the level homology and the full homology $H_*(\Lambda/\vartheta;\mathbb{Q})$ in one, as we will show.

- for n odd, $H_*(\Lambda/\vartheta;\mathbb{Q})$ is

$\uparrow d$						
$9n-8$					$\mathbb{Q}$	
$9n-9$						0
$8n-7$					0	
$8n-8$					$\mathbb{Q}$	
$7n-6$				$\mathbb{Q}$		
$7n-7$					0	
$6n-5$				0		
$6n-6$				$\mathbb{Q}$		
$5n-4$			$\langle q_{\vartheta *}(U^4)\rangle = \mathbb{Q}$			
$5n-5$				0		
$4n-3$			0			
$4n-4$			$\langle q_{\vartheta *}(A * U^4)\rangle = \mathbb{Q}$			
$3n-2$		$\langle q_{\vartheta *}(U^2)\rangle = \mathbb{Q}$				
$3n-3$			0			
$2n-1$		0				
$2n-2$		$\langle q_{\vartheta *}(A * U^2)\rangle = \mathbb{Q}$				
n	$\mathbb{Q}$					
$n-1$		0				
0	$\mathbb{Q}$					
	0	1	2	3	4	$5 \to r$

- for n even, $H_*(\Lambda/\vartheta;\mathbb{Q})$ is

$\uparrow d$						
$9n-8$					$\langle q_{\vartheta *}(\Theta^4)\rangle = \mathbb{Q}$	
$9n-9$						0
$8n-7$					0	
$8n-8$					0	
$7n-6$				0		
$7n-7$					$\langle q_{\vartheta *}(\sigma_2)\rangle = \mathbb{Q}$	
$6n-5$				0		
$6n-6$				0		
$5n-4$			$\langle q_{\vartheta *}(\Theta^2)\rangle = \mathbb{Q}$			
$5n-5$				0		
$4n-3$			0			
$4n-4$			0			
$3n-2$		0				
$3n-3$			$\langle q_{\vartheta *}(\sigma_2)\rangle = \mathbb{Q}$			
$2n-1$		0				
$2n-2$		0				
n	$\mathbb{Q}$					
$n-1$		0				
0	$\mathbb{Q}$					
	0	1	2	3	4	5 $\rightarrow r$

We give a elementary proof that this is also the homology of Λ/ϑ:

Proof of the Proposition. In this proof $H_*(\cdot)$ denotes singular homology with rational coefficients. We set $\Lambda^r/\vartheta := \Lambda S^{n\leq 2\pi^2 r^2}/\vartheta$ in this proof and thus

$$H_i(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta) = H_i(\Lambda S^{n\leq 2\pi^2 r^2}/\vartheta, \Lambda S^{n\leq 2\pi^2(r-1)^2}/\vartheta; \mathbb{Q}) \cong H_i(\Lambda S^{n\leq 2\pi^2 r^2}/\vartheta, \Lambda S^{n< 2\pi^2 r^2}/\vartheta; \mathbb{Q}).$$

The long exact homology sequence of the triple $(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta, \Lambda^0/\vartheta)$ is

$$\begin{tikzcd} \dots \arrow[r] & H_{k+1}(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta) \arrow[dlll, rounded corners, "\partial"'] & & \\ H_k(\Lambda^{r-1}/\vartheta, \Lambda^0/\vartheta) \arrow[r, "i"] & H_k(\Lambda^r/\vartheta, \Lambda^0/\vartheta) \arrow[r, "j"] & H_k(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta) \arrow[r] & \dots \end{tikzcd}$$

Fix $l \in \mathbb{N}$. Then there is at most one $r = r(l)$ with $H_l(\Lambda^{r(l)}/\vartheta, \Lambda^{r(l)-1}/\vartheta) \neq 0$ as the above diagrams show. We compute $H_l(\Lambda/\vartheta, \Lambda^0/\vartheta)$:

- If there is no such r, that is if $H_l(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta) = 0$ for all $r \in \mathbb{N}$, then the above sequence implies that

$$H_{l+1}(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta) \xrightarrow{\partial} H_l(\Lambda^{r-1}/\vartheta, \Lambda^0/\vartheta) \xrightarrow{i} H_l(\Lambda^r/\vartheta, \Lambda^0/\vartheta) \xrightarrow{j} 0$$

  is exact for all r, so i is surjective for all r. For $r = 1$ that is $0 = H_l(\Lambda^0/\vartheta, \Lambda^0/\vartheta) \twoheadrightarrow H_l(\Lambda^1/\vartheta, \Lambda^0/\vartheta)$, so $H_l(\Lambda^1/\vartheta, \Lambda^0/\vartheta) = 0$ and inductively

$$H_l(\Lambda^r/\vartheta, \Lambda^0/\vartheta) = 0$$

  for all r. Since the image of any singular (p-)simplex $\sigma : \Delta^p \to \Lambda S^n$ is contained in Λ^r for some r, taking the direct limit yields

$$H_l(\Lambda/\vartheta, \Lambda^0/\vartheta) = 0$$

  ([**Hat02**, Proposition 3.33]).
- If there is exactly one such $r(l)$ the same reasoning shows, via the exact sequence

$$H_{l+1}(\Lambda^s/\vartheta, \Lambda^{s-1}/\vartheta) \xrightarrow{\partial} H_l(\Lambda^{s-1}/\vartheta, \Lambda^0/\vartheta) \xrightarrow{i} H_l(\Lambda^s/\vartheta, \Lambda^0/\vartheta) \xrightarrow{j} 0,$$

  that $H_l(\Lambda^{s-1}/\vartheta, \Lambda^0/\vartheta) \xrightarrow{i} (\Lambda^s/\vartheta, \Lambda^0/\vartheta)$ is surjective $s < r(l)$. Thus

$$H_l(\Lambda^s/\vartheta, \Lambda^0/\vartheta) = 0$$

  for all $s < r(l)$. In particular $H_l(\Lambda^{r(l)-1}/\vartheta, \Lambda^0/\vartheta) = 0$. With $H_l(\Lambda^{r(l)}/\vartheta, \Lambda^{r(l)-1}/\vartheta) \cong \mathbb{Q}$ we get the following exact sequence:

$$0 \xrightarrow{i} H_l(\Lambda^r(l)/\vartheta, \Lambda^0/\vartheta) \xrightarrow{j} \mathbb{Q} \xrightarrow{\partial} H_{l-1}(\Lambda^{r(l)-1}/\vartheta, \Lambda^0/\vartheta).$$

  We now show that ∂ is the zero map and hence

$$H_l(\Lambda^r(l)/\vartheta, \Lambda^0/\vartheta) \cong \mathbb{Q}.$$

  - For n even this is easy: Since $H_k(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta)$ can only be nonzero if r is even, we know that the degree difference $|a - b|$ between any two nonzero level homology groups $H_a(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta)$ and $H_b(\Lambda^s/\vartheta, \Lambda^{s-1}/\vartheta)$
    * within the same energy level r is at least 7: $2n - 1 + \lambda_r - \lambda_r = 2n - 1 \geq 7$ since $n \geq 4$,
    * between different energy levels at least 5: $\lambda_{r+2} - (2n - 1 + \lambda_r) = 2n - 3 = 3n - 3 - n \geq 5$ since $n \geq 4$.

It follows that $H_{l-1}(\Lambda^{r-1}/\vartheta, \Lambda^{r-2}/\vartheta) = 0$ for all r and so $H_{l-1}(\Lambda^{r(l)-1}/\vartheta, \Lambda^0/\vartheta) = 0$.

– For n odd we have the degree difference is only bigger than 1 if we assume $n > 3$: The degree difference here is
  * within the energy level r at least 3: $2n - 1 + \lambda_r - (n - 1 + \lambda_r) = n \geq 3$ since $n \geq 3$,
  * between different energy levels at least 1: $n-1+\lambda_{r+1}-(2n-1+\lambda_r) = n-2$. For $n = 3$ that is only 1, otherwise at least 3.

For $n = 3$ we have to use the fact that $E_{g_{st}} : \Lambda S^n \to \mathbb{R}$ is perfect Morse-Bott function ([**GH09**, (proof of) Theorem 13.4] or [**Hin84**, Section 4]), the naturality of long exact homology sequence of pairs and that $q_* : H_k(\Lambda^r, \Lambda^{r-1}) \to H_k(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta)$ induced by the quotient map is sujective:

$$\begin{tikzcd}
H_l(\Lambda^{r(l)}, \Lambda^{r(l)-1}) \arrow[r, "\partial=0"] \arrow[d, "q_*"'] & H_{l-1}(\Lambda^{r(l)-1}, \Lambda^0) \arrow[d, "q_*"'] \\
H_l(\Lambda^{r(l)}/\vartheta, \Lambda^{r(l)-1}/\vartheta) \arrow[r, "\partial"'] & H_{l-1}(\Lambda^{r(l)-1}/\vartheta, \Lambda^0/\vartheta).
\end{tikzcd}$$

Hence the boundary map ∂ is also zero in the sequence at the bottom.

If $s > r(l)$ then $H_l(\Lambda^s/\vartheta, \Lambda^s/\vartheta) = 0 = H_{l+1}(\Lambda^s/\vartheta, \Lambda^{s-1}/\vartheta)$ since the degree difference is at least 5 if $n \geq 4$ or, more generally, ∂ is the zero map if $n \geq 3$. Thus i is an isomorphism since

$$\begin{tikzcd}
 & & & H_{l+1}(\Lambda^s/\vartheta, \Lambda^{s-1}/\vartheta) \arrow[dlll, "\partial=0"'] \\
H_l(\Lambda^{s-1}/\vartheta, \Lambda^0/\vartheta) \cong \mathbb{Q} \arrow[r, "i"] & H_l(\Lambda^s/\vartheta, \Lambda^0/\vartheta) \arrow[r, "j"] & H_l(\Lambda^s/\vartheta, \Lambda^{s-1}/\vartheta) = 0 &
\end{tikzcd}$$

is exact. This holds in particular for $s = r(l) + 1$ and hence

$$\mathbb{Q} \cong H_l(\Lambda^{r(l)}/\vartheta, \Lambda^0/\vartheta) \cong H_l(\Lambda^{r(l)+1}/\vartheta, \Lambda^0/\vartheta) \cong H_l(\Lambda/\vartheta, \Lambda^0/\vartheta)$$

by taking the direct limit.

Finally, for any subgroup $G \subset O(2)$ (acting in the obvious way on ΛM) the inclusion of constant curves $c : M \hookrightarrow \Lambda M^0 \subset \Lambda M$ and the evaluation $\Lambda M \to M \cong \Lambda M^0$ are both G-equivariant, hence $id_M = id_M/G = ev/G \circ c/G = ev/G \circ c$ and thus the exact sequence

$$0 \longrightarrow H_k(\Lambda^0/\vartheta) \xrightarrow{c_*} H_k(\Lambda/\vartheta) \longrightarrow H_k(\Lambda/\vartheta, \Lambda^0/\vartheta) \longrightarrow 0$$

splits to give $H_k(\Lambda/\vartheta) \cong H_k(\Lambda^0/\vartheta) \oplus H_k(\Lambda/\vartheta, \Lambda^0/\vartheta)$. The n-sphere only has homology in degrees 0 and n and in these degrees the level homology groups are trivial. □

REMARK 3.17. The above proof is actually a spectral sequence proof: The homology exact sequences of the triples $(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta, \Lambda^0/\vartheta)$ for different $r \in \mathbb{N}$ fit into the exact couple

$$\begin{tikzcd}
\bigoplus_{k,r\in\mathbb{N}} H_k(\Lambda^r/\vartheta, \Lambda^0/\vartheta) \arrow[rr, "i"] & & \bigoplus_{k,r\in\mathbb{N}} H_k(\Lambda^r/\vartheta, \Lambda^0/\vartheta) \arrow[dl, "j"] \\
& E^1 := \bigoplus_{k,r\in\mathbb{N}} H_k(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta) \arrow[ul, "\partial"] &
\end{tikzcd}$$

as described in [**Hat**, Chapter 1, Section 1]. The maps

$$d^1_{k+1,r} := j_{k,r-1} \circ \partial_{k+1,r} : E^1_{k+1,r} = H_{k+1}(\Lambda^r/\vartheta, \Lambda^{r-1}/\vartheta) \to E^1_{k,r-1} = H_k(\Lambda^{r-1}/\vartheta, \Lambda^{r-2}/\vartheta)$$

are then the differentials of the first page of the spectral sequence $E^1, E^2, \ldots$ which converges to $H_*(\Lambda, \Lambda^0)$. Above we have shown that

$$d^1_{k,r} = 0$$

for all $k, r \in \mathbb{N}$.

3.3. Computation of the algebra $\big(H_*(\Lambda S^n/\vartheta;\mathbb{Q}), P_\vartheta\big)$. We now compute the transfer product P_ϑ associated to orientation reversal. Recall that it is given by

$$P_\vartheta(a,b) = q_{\vartheta *}\big(tr_\vartheta(a) * tr_\vartheta(b)\big)$$

for $a, b \in H_*(\Lambda/\vartheta)$.

THEOREM 3.18. *Let $n > 2$ and let ϑ be the orientation reversal of loops on ΛS^n. Then*

- *for n odd, there exists a generator μ of $H_{3n-2}(\Lambda S^n/\vartheta;\mathbb{Q})$ which is not nilpotent in the algebra $(H_*(\Lambda S^n/\vartheta;\mathbb{Q}), P_\vartheta)$. More precisely, for every $k \in \mathbb{N}$, μ^k is a generator of $H_{2k(n-1)+n}(\Lambda S^n/\vartheta;\mathbb{Q})$.*
  Moreover, multiplication with μ, i.e.
  $$P_\vartheta(\cdot,\mu) : H_i(\Lambda S^n/\vartheta;\mathbb{Q}) \to H_{i+2n-2}(\Lambda S^n/\vartheta;\mathbb{Q})$$
  is an isomorphism for $i \geq 0$.
- *for n even, there exists a generator η of $H_{5n-4}(\Lambda S^n/\vartheta;\mathbb{Q})$ which is not nilpotent in the algebra $(H_*(\Lambda S^n/\vartheta;\mathbb{Q}), P_\vartheta)$. More precisely, for every $k \in \mathbb{N}$, η^k is a generator of $H_{4k(n-1)+n}(\Lambda S^n/\vartheta;\mathbb{Q})$.*
  Moreover, multiplication with η, i.e.
  $$P_\vartheta(\cdot,\eta) : H_i(\Lambda S^n/\vartheta;\mathbb{Q}) \to H_{i+4n-4}(\Lambda S^n/\vartheta;\mathbb{Q})$$
  is an isomorphism for $i > 0$.

PROOF. For the transfer tr_ϑ and the orbit map q_ϑ associated the the $\mathbb{Z}_2$-action ϑ we simply write tr and q in this proof. Also we just write Λ for ΛS^n. Recall the following properties of the transfer of a $\mathbb{Z}_2$-action:

(1) $(q_* \circ tr)(a) = 2a$,
(2) $(tr \circ q_*)(x) = x + \vartheta_*(x)$,
(3) $q_* : H_i(\Lambda;\mathbb{Q})^\vartheta := \{x \in H_i(\Lambda;\mathbb{Q}) \mid x = \vartheta_*(x)\} \to H_i(\Lambda/\mathbb{Z}_2;\mathbb{Q})$ is an isomorphism.

Hence we get

- for n odd, the generator U^{*2} of $H_{3n-2}(\Lambda;\mathbb{Q})$ is mapped to a generator μ of $H_{3n-2}(\Lambda/\vartheta;\mathbb{Q})$ via the quotient map $q:\Lambda\to\Lambda/\vartheta$. We now show that the class μ is nonnilpotent with respect to the product P_ϑ. We have
$$P_\vartheta(\mu,\mu)=q_*\big(tr(\mu)*tr(\mu)\big)=q_*\big((tr\circ q_*)(U^{*2})*(tr\circ q_*)(U^{*2})\big).$$
Since $(tr\circ q_*)(x)=x+\vartheta_*(x)$ for any $x\in H_*(\Lambda;\mathbb{Q})$
$$P_\vartheta(\mu,\mu)=q_*\big(U^{*4}+\vartheta_*(U^{*2})*U^{*2}+U^{*2}*\vartheta_*(U^{*2})+\vartheta_*(U^{*2})*\vartheta_*(U^{*2})\big)$$
follows. As $\vartheta_*(U^{*2})=\vartheta_*(U)^{*2}=(-U)^{*2}=U^{*2}$ we finally get
$$P_\vartheta(\mu,\mu)=4q_*(U^{*4}).$$
As $q_*(U^{*4})$ is a generator of $H_{5n-4}(\Lambda/\vartheta;\mathbb{Q})$, the class
$$\mu^2:=P_\vartheta(\mu,\mu)$$
is a generator of $H_{5n-4}(\Lambda/\vartheta;\mathbb{Q})$. The associativity of P_ϑ (see Proposition 2.2 in Chapter 3) now implies that μ^k is a generator of $H_{2n-1+\lambda_k}(\Lambda/\vartheta;\mathbb{Q})=H_{2k(n-1)+n}(\Lambda/\vartheta;\mathbb{Q})$ for every $k\in\mathbb{N}$.
In the same way one proves that $P(q_*(A),\mu^k)$ is nonzero for all $k\in\mathbb{N}$. We thus have that multiplication with μ is an isomorphism. It corresponds to jumping from one critical level to the next. Finally, one verifies that the element $q_*(E)$ is a unit up to scaling.

- for n even, the generator Θ^{*2} of $H_{5n-4}(\Lambda;\mathbb{Q})$ is mapped to a generator η of $H_{5n-4}(\Lambda/\vartheta;\mathbb{Q})$ via the quotient map $q:\Lambda\to\Lambda/\vartheta$. We have
$$P_\vartheta(\eta,\eta)=q_*\big(tr(\eta)*tr(\eta)\big)=q_*\big((tr\circ q_*)(\Theta^{*2})*(tr\circ q_*)(\Theta^{*2})\big).$$
With $(tr\circ q_*)(x)=x+\vartheta_*(x)$ for any $x\in H_*(\Lambda;\mathbb{Q})$ it follows that
$$P_\vartheta(\eta,\eta)=q_*\big(\Theta^{*4}+\vartheta_*(\Theta^{*2})*\Theta^{*2}+\Theta^{2}*\vartheta_*(\Theta^{*2})+\vartheta_*(\Theta^{*2})*\vartheta_*(\Theta^{*2})\big)$$
and since $\vartheta_*(\Theta^{*2})=\vartheta_*(\Theta)^{*2}=(-\Theta)^{*2}=\Theta^{*2}$ we finally get
$$P_\vartheta(\eta,\eta)=4q_*(\Theta^{*4}).$$
Again, $q_*(\Theta^{*4})$ is a generator of $H_{9n-8}(\Lambda/\vartheta;\mathbb{Q})$ and thus
$$\eta^2:=P_\vartheta(\eta,\eta)$$
is a generator of $H_{9n-8}(\Lambda/\vartheta;\mathbb{Q})$ and so, since P_ϑ is associative, η^k is a generator of $H_{2n-1+\lambda_{2k}}(\Lambda/\vartheta;\mathbb{Q})=H_{(4k+1)n-4k}(\Lambda/\vartheta;\mathbb{Q})$ for every $k\in\mathbb{N}$.
More generally, for arbitrary classes $y\in H_*(\Lambda/\vartheta;\mathbb{Q})$ we also have $q_*(x)=y$ for some $x\in H_*(\Lambda;\mathbb{Q})$. And hence
$$\begin{aligned}P_\vartheta(y,\eta)&=q_*\big((tr\circ q_*)(x)*(tr\circ q_*)(\Theta^{*2})\big)\\&=q_*\big(x*\Theta^{*2}+x*\vartheta_*(\Theta^{*2})+\vartheta_*(x)*\Theta^{*2}+\vartheta_*(x)*\vartheta_*(\Theta^{*2})\big)\end{aligned}$$
By the above properties we can choose x to be fixed under the action ϑ and so
$$\begin{aligned}P_\vartheta(y,\eta)&=q_*\big(x*\Theta^{*2}+x*\Theta^{*2}+x*\Theta^{*2}+x*\Theta^{*2}\big)\\&=4q_*(x*\Theta^{*2})\end{aligned}$$

As $\vartheta_*(x * \Theta^{*2}) = \vartheta_*(x) * \vartheta_*(\Theta^{*2}) = x * \Theta^{*2}$ we have $x * \Theta^{*2} \in H_{|x|+|\Theta^{*2}|-n}(\Lambda;\mathbb{Q})^\vartheta$ and hence

$$x * \Theta^{*2} \neq 0 \Leftrightarrow q_*(x * \Theta^{*2}) \neq 0.$$

□

With the fact that on even dimensional manifolds and with rational coefficients we have

$$A_\vartheta = P_\vartheta,$$

as shown in equation (4.7) of Chapter 3, an immediate corollary is

COROLLARY 3.19. *Let $n > 2$ be even and let ϑ be the orientation reversal of loops on ΛS^n. Then there exists a generator η of $H_{5n-4}(\Lambda/\vartheta;\mathbb{Q})$ which is not nilpotent in the algebra $(H_*(\Lambda S^n/\vartheta;\mathbb{Q}), A_\vartheta)$. More precisely, for every $k \in \mathbb{N}$, η^k is a generator of $H_{4k(n-1)+n}(\Lambda S^n/\vartheta;\mathbb{Q})$. Moreover, multiplication with η, i.e.*

$$A_\vartheta(\cdot,\eta) : H_i(\Lambda S^n/\vartheta;\mathbb{Q}) \to H_{i+4n-4}(\Lambda S^n/\vartheta;\mathbb{Q})$$

is an isomorphism for $i \neq 0$.

We now address the relation of the above to the action θ:

- for n odd we define the class $\nu \in H_{3n-3}(\Lambda/\theta;\mathbb{Q})$ via the class $\mu \in H_{3n-3}(\Lambda/\vartheta;\mathbb{Q})$:
$$\nu := \chi_*(\mu),$$
where χ_* is the homomorphism induced by the map $\chi_{\frac{1}{4}} : \Lambda S^n \to \Lambda S^n$. By Theorem 2.7, Section 2 of Chapter 3, χ_* is an algebra isomorphism $\chi_* : \big(H_*(\Lambda/\vartheta;\mathbb{Q}), P_\vartheta\big) \to \big(H_*(\Lambda/\theta;\mathbb{Q}), P_\theta\big)$. It follows that in the above theorem we can simply replace ϑ by θ and μ by ν and get the same theorem for the algebra $\big(H_*(\Lambda S^n/\theta;\mathbb{Q}), P_\theta\big)$.
- for n even we choose to denote the generator of $H_{5n-4}(\Lambda S^n/\theta;\mathbb{Q})$ corresponding under χ_* to η also by η. This is justified by noting the following: As we saw above, for spheres the loop space homology comes from the homology of the critical submanifolds of great circles B_r. On these submanifolds the two actions ϑ and θ are related in the following way: For $\gamma_r \in B_r$ we have
$$\theta(\gamma_r) = \begin{cases} \vartheta(\gamma_r), & \text{if } r \text{ is even} \\ (\vartheta \circ \chi_{\frac{1}{2}})(\gamma_r), & \text{if } r \text{ is odd} \end{cases}.$$
For even spheres $q_{\theta *}$ and $q_{\vartheta *}$ are both trivial on classes coming from B_r with r odd and equal on classes coming from B_r with r even.

This proves Theorem 2.1 of the last section.

REMARK 3.20. Summarizing the above theorems and discussion we have

- with rational coefficients $H_*(\Lambda/\vartheta;\mathbb{Q})$ is just that part of the homology $H_*(\Lambda;\mathbb{Q})$ that is invariant under the involution ϑ_*.
- for $\mathbb{Q}$-coefficients and n odd $A_\vartheta = 0$ but P_ϑ is a nontrivial product on $H_*(\Lambda S^n/\vartheta;\mathbb{Q})$, the latter corresponds to the restriction of the Chas-Sullivan product to classes invariant under ϑ_*.
- for $\mathbb{Q}$-coefficients and n even $A_\vartheta = P_\vartheta$ is a nontrivial product on $H_*(\Lambda S^n/\vartheta;\mathbb{Q})$, it corresponds to the restriction of the Chas-Sullivan product to classes invariant under ϑ_*.
- P_ϑ are P_θ essentially the same.
- for coefficients other than $\mathbb{Q}$ we have not yet computed P_ϑ, P_θ, A_ϑ, A_θ.

4. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-action $\vartheta \times \theta$ on spheres: Computation of $\big(H_*(\Lambda S^n/(\vartheta \times \theta); \mathbb{Q}), P_{\vartheta\times\theta}\big)$

Let S^n be endowed with the standard Riemannian metric.

We now consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action $\vartheta \times \theta$ on $\Lambda = \Lambda S^n$ which is given by the three involutions

$$\begin{aligned}\vartheta(\gamma) &:= \bar{\gamma}\\ \theta(\gamma) &:= \frac{1}{2}.\bar{\gamma} = \overline{\frac{1}{2}.\gamma}\\ (\vartheta \times \theta)(\gamma) &:= (\vartheta \circ \theta)(\gamma) = (\theta \circ \vartheta)(\gamma) = \chi_{\frac{1}{2}}(\gamma) = \frac{1}{2}.\gamma\end{aligned}$$

for $\gamma \in \Lambda$. This last involution has fixed points within the set of closed geodesics! An even iterate of a closed geodesic is mapped to itself under $\chi_{\frac{1}{2}}$. The action is thus not free on B_r if r is even.
We therefore have to use equivariant homology and study the $\mathbb{Z}_2 \times \mathbb{Z}_2$-vector bundle

$$p_r \times id : \Gamma_r^- \times E(\mathbb{Z}_2 \times \mathbb{Z}_2) \to B_r \times E(\mathbb{Z}_2 \times \mathbb{Z}_2).$$

Since the action on the base $B_r \times E(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is now free, we get vector bundles

$$(p_r \times id)/(\vartheta \times \theta) : \Gamma_r^- \times_{\mathbb{Z}_2\times\mathbb{Z}_2} E(\mathbb{Z}_2 \times \mathbb{Z}_2) \to B_r \times_{\mathbb{Z}_2\times\mathbb{Z}_2} E(\mathbb{Z}_2 \times \mathbb{Z}_2).$$

We have to check whether these bundles are orientable and we have to compute their homology relative to the zero section

$$H_i^{\mathbb{Z}_2\times\mathbb{Z}_2}(\Lambda^{\le 2\pi^2 r}, \Lambda^{<2\pi^2 r}; R) \cong H_i^{\mathbb{Z}_2\times\mathbb{Z}_2}(D\Gamma_r^-, \partial D\Gamma_r^-; R) \cong H_{i-\lambda_r}^{\mathbb{Z}_2\times\mathbb{Z}_2}(B_r; R),$$

where the last equation only holds if the bundle is R-orientable. Since $\vartheta_* = \theta_*$ and $(\chi_{\frac{1}{2}})_* = id$, by Corollaries 3.2 and 3.8 of the last section we know already that this is the case if and only if n and r are even.

Since the action $\vartheta \times \theta$ is free on B_r if r is odd we can distinguish the two cases:

- if r is odd, we have

  $$H_{i-\lambda_r}^{\mathbb{Z}_2\times\mathbb{Z}_2}(B_r; R) \cong H_{i-\lambda_r}\big(B_r/(\vartheta \times \theta); R\big)$$

  induced by the fibre bundle projection $\pi : B_r \times_{\mathbb{Z}_2\times\mathbb{Z}_2} E(\mathbb{Z}_2 \times \mathbb{Z}_2) \to B_r/(\vartheta \times \theta)$. This is a fibre bundle because the action on B_r is free. Its fibre $E(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is contractible and hence, by the Leray-Hirsch theorem, π is a weak homotopy equivalence.
  Freeness of the action on B_r implies that the action is also free on Γ_r^-, hence, the same reasoning yields

  $$\begin{aligned}H_i^{\mathbb{Z}_2\times\mathbb{Z}_2}(D\Gamma_r^-, \partial D\Gamma_r^-; R) &\cong H_i\big(D\Gamma_r^-/(\vartheta \times \theta), \partial D\Gamma_r^-/(\vartheta \times \theta); R\big)\\ &\cong H_i\big(\Gamma_r^-/(\vartheta \times \theta), (\Gamma_r^- - B_r)/(\vartheta \times \theta); R\big).\end{aligned}$$

  It follows that

  $$H_i^{\mathbb{Z}_2\times\mathbb{Z}_2}(\Lambda^{\le 2\pi^2 r}, \Lambda^{<2\pi^2 r}; \mathbb{Q}) \cong H_i\big(\Lambda^{\le 2\pi^2 r}/(\vartheta \times \theta), \Lambda^{<2\pi^2 r}/(\vartheta \times \theta); \mathbb{Q}\big).$$

Since r is odd the bundle $p_r/(\vartheta \times \theta) : \Gamma_r^-/(\vartheta \times \theta) \to B_r/(\vartheta \times \theta)$ is not orientable and, using the transfer properties (Section 2 of Chapter 3) again, we get

$$\begin{aligned} H_i\big(\Lambda^{\le 2\pi^2 r}/(\vartheta \times \theta), \Lambda^{<2\pi^2 r}/(\vartheta \times \theta); \mathbb{Q}\big) &\cong H_i\big(\Gamma_r^-/(\vartheta \times \theta), (\Gamma_r^- - B_r)/(\vartheta \times \theta); \mathbb{Q}\big) \\ &\cong H_{i-\lambda_r}(B_r; \mathbb{Q})/H_{i-\lambda_r}(B_r; \mathbb{Q})^{\vartheta \times \theta} \\ &\cong H_{i-\lambda_r}(B_r; \mathbb{Q})/H_{i-\lambda_r}(B_r; \mathbb{Q})^{\theta} \\ &\cong H_{i-\lambda_r}(B_r; \mathbb{Q})/H_{i-\lambda_r}(B_r; \mathbb{Q})^{\vartheta} \\ &\cong H_i(\Lambda^{\le 2\pi^2 r}/\vartheta, \Lambda^{<2\pi^2 r}/\vartheta; \mathbb{Q}). \end{aligned}$$

Since $\vartheta_* = \theta_*$ and $(\chi_{\frac{1}{2}})_* = id$, the spaces again decompose into ± 1-eigenspaces.

- if r is even, the orbit of an element γ_r of B_r under the action $\vartheta \times \theta$ is

$$\begin{aligned} \{\gamma_r, \vartheta(\gamma_r), \theta(\gamma_r), (\vartheta \circ \theta)(\gamma_r)\} &= \{\gamma_r, \vartheta(\gamma_r), (\vartheta \circ \chi_{\frac{1}{2}})(\gamma_r), (\vartheta \circ \vartheta \circ \chi_{\frac{1}{2}})(\gamma_r)\} \\ &= \{\gamma_r, \vartheta(\gamma_r), \vartheta(\gamma_r), \gamma_r\} \\ &= \{\gamma_r, \vartheta(\gamma_r)\} \end{aligned}$$

and so $\vartheta \times \theta$ makes the same identifications as ϑ on B_r. It follows that

$$B_r/\vartheta \cong B_r/(\vartheta \times \theta).$$

In fact the normal subgroup $\{id, \vartheta \circ \theta\} \subset \{id, \vartheta, \theta, \vartheta \circ \theta\}$ corresponding to the diagonal embedding $\Delta : \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ is the isotropy group of each element of B_r. We now use that we also have an oriented and an unoriented version of the Thom isomorphism for the vector bundles $\Gamma_r^- \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} E(\mathbb{Z}_2 \times \mathbb{Z}_2) \to B_r \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} E(\mathbb{Z}_2 \times \mathbb{Z}_2)$. We also use the homological Künneth formula and that $E(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is contractible. Hence all induced maps are the identity on $H_0\big(E(\mathbb{Z}_2 \times \mathbb{Z}_2)\big)$ and we have

  - in the nonorientable case, i.e. if n is odd

$$\begin{aligned} H_i^{\mathbb{Z}_2 \times \mathbb{Z}_2}(\Lambda^{\le 2\pi^2 r}, \Lambda^{<2\pi^2 r}; \mathbb{Q}) &\cong H_i\big(\Gamma_r^- \times_{\vartheta \times \theta} E(\mathbb{Z}_2 \times \mathbb{Z}_2), (\Gamma_r^- - B_r) \times_{\vartheta \times \theta} E(\mathbb{Z}_2 \times \mathbb{Z}_2); \mathbb{Q}\big) \\ &\cong H_i\big(\Gamma_r^- \times E(\mathbb{Z}_2 \times \mathbb{Z}_2), (\Gamma_r^- - B_r) \times E(\mathbb{Z}_2 \times \mathbb{Z}_2); \mathbb{Q}\big)^{\vartheta \times \theta} \\ &\cong H_{i-\lambda_r}(B_r \times E(\mathbb{Z}_2 \times \mathbb{Z}_2); \mathbb{Q})/H_{i-\lambda_r}(B_r \times E(\mathbb{Z}_2 \times \mathbb{Z}_2); \mathbb{Q})^{\vartheta \times \theta} \\ &\cong H_{i-\lambda_r}(B_r; \mathbb{Q})/H_{i-\lambda_r}(B_r; \mathbb{Q})^{\vartheta \times \theta} \\ &\cong H_{i-\lambda_r}(B_r; \mathbb{Q})/H_{i-\lambda_r}(B_r; \mathbb{Q})^{\theta} \\ &\cong H_{i-\lambda_r}(B_r; \mathbb{Q})/H_{i-\lambda_r}(B_r; \mathbb{Q})^{\vartheta} \\ &\cong H_i(\Lambda^{\le 2\pi^2 r}/\vartheta, \Lambda^{<2\pi^2 r}/\vartheta; \mathbb{Q}). \end{aligned}$$

– in the orientable case, i.e. if n is even

$$\begin{aligned}
H_i^{\mathbb{Z}_2\times\mathbb{Z}_2}(\Lambda^{\le 2\pi^2 r},\Lambda^{<2\pi^2 r};\mathbb{Q}) &\cong H_i\big(\Gamma_r^-\times_{\vartheta\times\theta}E(\mathbb{Z}_2\times\mathbb{Z}_2),(\Gamma_r^- - B_r)\times_{\vartheta\times\theta}E(\mathbb{Z}_2\times\mathbb{Z}_2);\mathbb{Q}\big)\\
&\cong H_i\big(\Gamma_r^-\times E(\mathbb{Z}_2\times\mathbb{Z}_2),(\Gamma_r^- - B_r)\times E(\mathbb{Z}_2\times\mathbb{Z}_2);\mathbb{Q}\big)^{\vartheta\times\theta}\\
&\cong H_{i-\lambda_r}(B_r\times E(\mathbb{Z}_2\times\mathbb{Z}_2);\mathbb{Q})^{\vartheta\times\theta}\\
&\cong H_{i-\lambda_r}(B_r;\mathbb{Q})^{\vartheta\times\theta}\\
&\cong H_{i-\lambda_r}(B_r;\mathbb{Q})^{\theta}\\
&\cong H_{i-\lambda_r}(B_r;\mathbb{Q})^{\vartheta}\\
&\cong H_i(\Lambda^{\le 2\pi^2 r}/\vartheta,\Lambda^{<2\pi^2 r}/\vartheta;\mathbb{Q}).
\end{aligned}$$

Moreover, in both cases (n odd or even)

$$\begin{aligned}
H_i(\Lambda^{\le 2\pi^2 r}/(\vartheta\times\theta),\Lambda^{<2\pi^2 r}/(\vartheta\times\theta);\mathbb{Q}) &\cong H_i\big(\Gamma_r^-/(\vartheta\times\theta),(\Gamma_r^- - B_r)/(\vartheta\times\theta);\mathbb{Q}\big)\\
&\cong H_i\big(\Gamma_r^-,\Gamma_r^- - B_r;\mathbb{Q}\big)^{\vartheta\times\theta}\\
&\cong H_i\big(\Gamma_r^-\times E(\mathbb{Z}_2\times\mathbb{Z}_2),(\Gamma_r^- - B_r)\times E(\mathbb{Z}_2\times\mathbb{Z}_2);\mathbb{Q}\big)^{\vartheta\times\theta}\\
&\cong H_i(\Lambda^{\le 2\pi^2 r}/\vartheta,\Lambda^{<2\pi^2 r}/\vartheta;\mathbb{Q}).
\end{aligned}$$

Therefore

THEOREM 4.1. *For the homologies we have*

$$H_*(\Lambda S^n/\theta;\mathbb{Q})\cong H_*(\Lambda S^n/\vartheta;\mathbb{Q})\cong H_*(\Lambda S^n/(\vartheta\times\theta);\mathbb{Q})$$

as graded vector spaces. On rational homology the transfer products $P_\theta, P_\vartheta, P_{\vartheta\times\theta}$ differ at most by scaling.

□

5. $\mathbb{Z}_2$-Equivariant homology on spheres with $\mathbb{Z}_2$-coefficients

As observed earlier, if given an arbitrary $\mathbb{Z}_2$-action f on ΛS^n for which we would like to compute the equivariant loop bracket

$$B: H_i^{\mathbb{Z}_2}(\Lambda S^n)\times H_j^{\mathbb{Z}_2}(\Lambda S^n)\to H_{i+j-n}^{\mathbb{Z}_2}(\Lambda S^n)$$

defined by

$$\begin{tikzcd}
H_i^{\mathbb{Z}_2}(\Lambda S^n) \times H_j^{\mathbb{Z}_2}(\Lambda S^n) \arrow[d, "\partial\circ\text{Thom isom.}\times\partial\circ\text{Thom isom.}"'] \\
H_i(\Lambda S^n \times E\mathbb{Z}_2) \times H_j(\Lambda S^n \times E\mathbb{Z}_2) \arrow[d, "\cong"'] \\
H_i(\Lambda S^n) \times H_j(\Lambda S^n) \arrow[d, "\text{Chas - Sullivan product}"'] \\
H_{i+j-n}(\Lambda S^n) \arrow[d, "\cong"'] \\
H_{i+j-n}(\Lambda S^n \times E\mathbb{Z}_2) \arrow[d, "p_*"'] \\
H_{i+j-n}^{\mathbb{Z}_2}(\Lambda S^n).
\end{tikzcd}$$

We need to take homology with $\mathbb{Z}_2$-coefficients since $p : \Lambda S^n \times E\mathbb{Z}_2 \to \Lambda S^n \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ is a nonorientable S^0-bundle if $n > 2$.

We are not going to compute B for spheres, but we indicate below, that it is computationally more challenging than the above computations with rational coefficients: In order to get a product on level homologies induced by the product B, we consider actions f which leave the energy invariant. Critical submanifolds $B \subset \Lambda$ are then equivariantly embedded. Let us look at the action ϑ, which is free on the space of great circles B_r for all r: We again have

$$H_{i-\lambda_r}^{\mathbb{Z}_2}(B_r;\mathbb{Z}_2) = H_{i-\lambda_r}(B_r \times_{\mathbb{Z}_2} E\mathbb{Z}_2;\mathbb{Z}_2) \cong H_{i-\lambda_r}(B_r/\vartheta;\mathbb{Z}_2)$$

since the action is free and

$$H_i^{\mathbb{Z}_2}(\Lambda S^{n\le 2\pi^2 r}, \Lambda^{<2\pi^2 r};\mathbb{Z}_2) \cong H_{i-\lambda_r}^{\mathbb{Z}_2}(B_r;\mathbb{Z}_2)$$

since the bundles $p_r : \Gamma_r^- \to B_r$ are always $\mathbb{Z}_2$-orientable. Recall that $\lambda_r = r\lambda_1 + (r-1)(n-1) = (2r-1)(n-1)$. We have

$$H_i(B_r/\vartheta;\mathbb{Z}_2) \cong H_i(T^1S^n/\vartheta;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i = 0,\dots,2n-1 \\ 0, & \text{otherwise} \end{cases}.$$

To see this, recall that the involution ϑ on $T^1S^n \cong V_2(\mathbb{R}^{n+1})$ is given by $(p,v) \mapsto (p-v)$. It follows that B_r/ϑ is the total space of an $\mathbb{R}P^{n-1}$-bundle over S^n: $\mathbb{R}P^{n-1} \to B_r/\vartheta \xrightarrow{p} S^n$. Its long exact homotopy sequence is

$$\dots \to \pi_2(S^n) \to \pi_1(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \pi_1(B_r/\vartheta) \xrightarrow{p_*} \pi_1(S^n) \to \dots$$

For $n \ge 3$ the outermost groups are trivial, hence the homomorphism i_*, which is induced by the inclusion i of the fibre, is an isomorphism. The two 2-sheeted covering maps $S^{n-1} \to \mathbb{R}P^{n-1}$ and $B_r \to B_r/\vartheta$ imply that $\pi_1(\mathbb{R}P^{n-1}) \cong \mathbb{Z}_2 \cong \pi_1(B_r/\vartheta)$ for $n \ge 3$, because the total spaces are simply-connected. For integral homology we thus have the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_2 \cong \pi_1(\mathbb{R}P^{n-1}) \arrow[r, "i_*", "\cong"'] \arrow[d, "\cong"', "h"] & \pi_1(B_r/\vartheta) \cong \mathbb{Z}_2 \arrow[d, "\cong"', "h"] \\
H_1(\mathbb{R}P^{n-1}) \arrow[r, "i_*"] & H_1(B_r/\vartheta)
\end{tikzcd}$$

where h is the Hurewicz map. Hence the first integral homology groups of $\mathbb{R}P^{n-1}$ and B_r/ϑ are both isomorphic to $\mathbb{Z}_2$. For $\mathbb{Z}_2$-coefficients we have

$$H^k(\mathbb{R}P^{n-1}; \mathbb{Z}_2) = H_k(\mathbb{R}P^{n-1}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0, \dots, n-1 \\ 0, & \text{otherwise} \end{cases}$$

and we know the cup ring is a truncated polynomial ring: $\big(H^*(\mathbb{R}P^{n-1}; \mathbb{Z}_2), \cup\big) \cong \mathbb{Z}_2[y]/(y^n)$ where y is a class of degree 1 ([**Hat02**, Theorem 3.19]). We set $c_1 := i^*(y)$ and $c_j := c^{\cup j} \in H^j\big(B_r/\vartheta, \mathbb{Z}_2\big)$. Then, for $0 \leq j < n$, we have $i^*(c_j) = i^*(c^{\cup j}) = i^*(c_1)^{\cup j} = y^{\cup j} \neq 0$. We thus have cohomology classes c_j of B_r/ϑ that restrict to a $\mathbb{Z}_2$-basis of the homology for each fibre of the bundle $\mathbb{R}P^{n-1} \to B_r/\vartheta \xrightarrow{p} S^n$. The Leray-Hirsch theorem ([**Hat02**, Theorem 4D.1]) is thus applicable and gives

$$H^*\big(B_r/\vartheta; \mathbb{Z}_2\big) \cong H^*(S^n; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(\mathbb{R}P^{n-1}; \mathbb{Z}_2).$$

The $\mathbb{Z}_2$-modules $H^*\big(B_r/\vartheta; \mathbb{Z}_2\big)$ and $H^*(\mathbb{R}P^{2n-1}; \mathbb{Z}_2)$ are therefore isomorphic

$$H^i\big(B_r/\vartheta; \mathbb{Z}_2\big) \cong \begin{cases} \mathbb{Z}_2, & i = 0, \dots, 2n-1 \\ 0, & \text{otherwise} \end{cases}$$

Hence B_r/ϑ is a "cohomology-$\mathbb{R}P^{2n-1}$", at least additively and with $\mathbb{Z}_2$-coefficients.

Thus if we plot the $\mathbb{Z}_2$-equivariant level homology with $H_d^{\mathbb{Z}_2}(\Lambda S^{n \leq 2\pi^2 r}, \Lambda S^{n < 2\pi^2 r}; \mathbb{Z}_2)$ at the coordinate (r, d) we get

$\uparrow d$				
$6n-5$	$\mathbb{Z}_2$			$\mathbb{Z}_2$
$6n-6$	$\mathbb{Z}_2$			$\mathbb{Z}_2$
	$\vdots$			$\vdots$
$5n-4$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
$5n-5$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\vdots$		$\vdots$	
$4n-3$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	
$4n-4$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	
	$\vdots$		$\vdots$	
$3n-2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
$3n-3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
	$\vdots$	$\vdots$		
$2n-1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
$2n-2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
	$\vdots$	$\vdots$		
n	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
$n-1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
	$\vdots$			
0	$\mathbb{Z}_2$			
	0	1	2	$3 \ \to r$

where for $r = 0$ we have $H_i^{\mathbb{Z}_2}(\Lambda S^{n \leq 2\pi^2 0}, \Lambda^{<2\pi^2 0}) = H_i^{\mathbb{Z}_2}(\Lambda S^{n=0}, \emptyset) = H_i^{\mathbb{Z}_2}(\Lambda S^{n0})$. $\mathbb{Z}_2$ acts trivially on $S^n \cong \Lambda S^{n0}$ hence

$$H_i^{\mathbb{Z}_2}(\Lambda S^{n0}) = H_i(\Lambda S^{n0} \times_{\mathbb{Z}_2} E\mathbb{Z}_2) \cong H_i(S^n \times_{\mathbb{Z}_2} E\mathbb{Z}_2) \cong H_i(S^n \times B\mathbb{Z}_2) = H_i(S^n \times \mathbb{R}P^\infty).$$

In contrast to the the previous section, we now have a lot of homology. The above diagram shows the first page of a spectral sequence that might not degenerate at the first page. We cannot compute the homology $H_*^{\mathbb{Z}_2}(\Lambda; \mathbb{Z}_2)$ without further information.

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